A new spectral cancellation in quantum gravity

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Plan of the talk

1. Physical motivations
2. Boundary conditions
3. Eigenvalue condition for scalar modes
4. Four ζ-functions for scalar modes
5. Regularity of ζ_B^+ at $s = 0$
6. Open problems

1. Physical motivations

(i) Functional integrals and space-time view
(ii) The whole set of physical laws from invariance principles

1. Physical motivations

(iii) Early universe from quantum physics; wave function of the universe and Hartle–Hawking quantum cosmology; gravitational instantons
(iv) Spectral theory; functional determinants in one-loop quantum theory; first corrections to classical dynamics

2. Boundary conditions

Unified scheme for Maxwell, YM, GR:

(1) $$\Big[\pi\mathcal{A}\Big]_{\mathcal{B}} = 0,$$

(2) $$\Big[\Phi(\mathcal{A})\Big]_{\mathcal{B}} = 0,$$

(3) $$\Big[\varphi\Big]_{\mathcal{B}} = 0.$$

Here π is a projector acting on the gauge field $\mathcal{A}$, Φ is the gauge-fixing functional, φ the ghost field.

2. Boundary conditions

Both (1) and (2) are preserved under infinitesimal gauge transformations provided that the ghost obeys homogeneous Dirichlet conditions as in (3). For gravity, we choose Φ so as to have an operator P of Laplace type in the Euclidean theory.

3. Eigenvalue condition for scalar modes

On the Euclidean 4-ball, we expand metric perturbations $h_{\mu\nu}$ in terms of scalar, transverse vector, transverse-traceless tensor harmonics on S^3. For vector, tensor and ghost modes, boundary conditions reduce to Dirichlet or Robin. For scalar modes, one finds eventually the eigenvalues $E = x^2$ from the roots x of

3. Eigenvalue condition for scalar modes

(4) $$J_n'(x) \pm \frac{n}{x} J_n(x) = 0,$$

(5) $$J_n'(x) + \left(-\frac{x}{2} \pm \frac{n}{x}\right) J_n(x) = 0.$$

Note that both x and $-x$ solve the same equation.

4. Four ζ-functions for scalar modes

(6) $$\zeta^{\pm}_{A,B}(s) \equiv \frac{(\sin \pi s)}{\pi} \sum_{n=3}^{\infty} n^{-(2s-2)} \int_0^{\infty} dz \, \frac{\frac{\partial}{\partial z} \log F^{\pm}_{A,B}(zn)}{z^{2s}}$$

where (here $\beta_+ \equiv n, \beta_- \equiv n+2$)

(7) $$F^{\pm}_{A}(zn) \equiv z^{-\beta_{\pm}} \Big(zn I'_n(zn) \pm n I_n(zn) \Big),$$

4. Four ζ-functions for scalar modes

$$F_B^{\pm}(zn) \equiv z^{-\beta_{\pm}} \left(zn I_n'(zn) + \left(\frac{z^2 n^2}{2} \pm n \right) I_n(zn) \right) . \tag{8}$$

Regularity at the origin is easily proved in the elliptic sectors, corresponding to $\zeta_A^{\pm}(s)$ and $\zeta_B^{-}(s)$.

5. Regularity of ζ_B^+ at $s = 0$

We define $\tau \equiv (1+z^2)^{-1/2}$ and consider the uniform asymptotics

(9) $$F_B^+(zn) \sim \frac{e^{n\eta(\tau)}}{h(n)\sqrt{\tau}} \frac{(1-\tau^2)}{\tau} \left(1 + \sum_{j=1}^{\infty} \frac{r_{j,+}(\tau)}{n^j}\right).$$

On splitting $\int_0^1 d\tau = \int_0^\mu d\tau + \int_\mu^1 d\tau$ with μ small, we get an asymptotic expansion of the l.h.s. by writing, *in the first interval* on the r.h.s.,

5. Regularity of ζ_B^+ at $s = 0$

(10)
$$\log\left(1+\sum_{j=1}^{\infty}\frac{r_{j,+}(\tau)}{n^j}\right)\sim\sum_{j=1}^{\infty}\frac{R_{j,+}(\tau)}{n^j},$$

and then computing

(11)
$$C_j(\tau)\equiv\frac{\partial R_{j,+}}{\partial\tau}=(1-\tau)^{-j-1}\sum_{a=j-1}^{4j}K_a^{(j)}\tau^a.$$

5. Regularity of ζ_B^+ at $s = 0$

Remarkably, by virtue of the identity

(12) $$g(j) \equiv \sum_{a=j}^{4j} \frac{\Gamma(a+1)}{\Gamma(a-j+1)} K_a^{(j)} = 0,$$

which holds $\forall j = 1, \ldots, \infty$, we find

(13) $$\lim_{s \rightarrow 0} s \zeta_B^+(s) = \frac{1}{6} \sum_{a=3}^{12} a(a-1)(a-2) K_a^{(3)} = 0,$$

5. Regularity of ζ_B^+ at $s = 0$

and

$$\begin{aligned} \zeta_B^+(0) &= \frac{5}{4} + \frac{1079}{240} - \frac{1}{2}\sum_{a=2}^{12}\omega(a)K_a^{(3)} \\ &+ \sum_{j=1}^{\infty} f(j)g(j) = \frac{296}{45}, \end{aligned} \tag{14}$$

where

5. Regularity of ζ_B^+ at $s = 0$

$$\begin{aligned}
\omega(a) \quad &\equiv \quad \frac{1}{6}\frac{\Gamma(a+1)}{\Gamma(a-2)}\Big[-\log(2) \\
&- \quad \frac{(6a^2-9a+1)}{4}\frac{\Gamma(a-2)}{\Gamma(a+1)} \\
&+ \quad 2\psi(a+1)-\psi(a-2)-\psi(4)\Big],
\end{aligned} \tag{15}$$

5. Regularity of ζ_B^+ at $s = 0$

$$
\begin{aligned}
f(j) &\equiv \frac{(-1)^j}{j!}\Big[-1-2^{2-j} \\
&+ \zeta_R(j-2)(1-\delta_{j,3})+\gamma\delta_{j,3}\Big].
\end{aligned} \tag{16}
$$

Equation (12) achieves 3 goals:

(i) Vanishing of $\log(2)$ coefficient in (14)
(ii) Vanishing of $\sum_{j=1}^{\infty} f(j)g(j)$ in (14)
(iii) Regularity at the origin of ζ_B^+

6. Open problems

(i) Cross-check of our analysis without using contour integration (in progress).
(ii) Foundations of Eq. (12): if a sector of $\mathbf{C}$ exists which is free of eigenvalues of the leading symbol of P, one can define complex powers of P even though the heat operator does not exist.

6. Open problems

(iii) We might instead consider non-local boundary data, e.g. those giving rise to surface states for the Laplacian.
(iv) Possible relevance for AdS/CFT in light of a profound link between AdS/CFT and the Hartle–Hawking wave function of the universe.

Recent References

G. Esposito, G. Fucci, A.Yu. Kamenshchik, K. Kirsten, CQGRD,22,957-974 (2005). [HEP-TH/0412269]
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