EUCLIDEAN QUANTUM GRAVITY:

FIELD THEORETICAL AND COSMOLOGICAL ASPECTS

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1. PHYSICAL MOTIVATIONS

1.1 Unification of guiding principles (quantum physics and general relativity)

1.2 Unified picture of theoretical physics of fundamental interactions

1.3 Cosmological singularities

1.4 Physics at the Planck scale

2. LAGRANGIAN ROAD

2.1 Manifestly covariant:

2.1a Background field method, to maintain manifest covariance in QFT despite the introduction of special gauges to secure well-defined propagators.

2.1b Ghost fields in the quantization of Yang–Mills and Einstein theory.

2.1c Applications: scattering of a graviton by a material particle; gravitational radiation from accelerating masses; emission of gravitons and infrared problem.

2.2 Lorentzian and Euclidean functional integrals

2.2a Space-time (M, g), space of field histories and

$$\psi_\sigma(f_\sigma) = \int K\Big(f_\sigma, \sigma; f_0, \sigma_0\Big)\ \psi_0(f_0)\ \mu(f_0),$$

$$K_L\Big(f_2, \sigma_2; f_1, \sigma_1\Big) = \int_{C_L} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} I(f)}\ \mu(f),$$

$$K_E\Big(f_2, \sigma_2; f_1, \sigma_1\Big) = \int_{C_R} \mathrm{e}^{-\frac{I_E(f)}{\hbar}}\ \mu_E(f) = Z[\text{boundary data}].$$

2.3 Higher-derivative theories (they are perturbatively renormalizable, but lead to negative-norm states and loss of unitarity)

2.4 Supergravity

2.5 Strings and brane world

3. DEEPER LOOK AT EUCLIDEAN QUANTUM GRAVITY

3.1 Gravitational instantons: complete non-singular Riemannian 4-geometries (M, g) with positive-definite 4-metric g solving the Riemannian Einstein equations (here $\Lambda = 0$ or $\Lambda \neq 0$)

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 0 \Longrightarrow R_{ab} = \Lambda g_{ab}.$$

3.1a Asymptotically Euclidean: only flat E_4.

3.1b Asymptotically locally Euclidean: the boundary at infinity has topology S^3/Γ, with Γ a discrete sub-group of $SO(4)$ (Eguchi–Hanson; multi-instantons).

3.1c Asymptotically flat: the 4-metric tends to the flat metric in 3 directions, but is periodic in the Euclidean-time variable (Schwarzschild).

3.1d Asymptotically locally flat: unlike 3.1b, the S^3 is distorted and expands with increasing radius in only 2 directions rather than 3 (2-parameter Taub-NUT metrics; multi-Taub-NUT instantons).

3.1e Compact: here $\Lambda \neq 0$, and one deals with $S^4; CP^2; S^2 \times S^2$; S^2 bundle over S^2; $K3$ surface.

3.2 Partition functions and geometrical aspects of thermodynamics

$$Z = \mathrm{Tr}\, \mathrm{e}^{-\beta H} = \int \mathrm{d}[\phi] \mathrm{e}^{\mathrm{i}I[\phi]}$$

is a functional integral over all fields periodic with period β in imaginary time.

Now for pure gravity (Schwarzschild) we write $g = g_0 + \gamma$ and we find

$$\log Z \sim \mathrm{i}I[g_0] + \log \int \mathrm{d}[\gamma] \mathrm{e}^{\mathrm{i}I_2[\gamma]} + \text{higher order terms},$$

and hence (κ being the surface gravity)

$$\begin{aligned} W &= \text{thermodynamic potential} \\ &= -T \log Z \sim -\mathrm{i}TI[g_0] = -\mathrm{i}\frac{\kappa}{2\pi}\frac{\mathrm{i}\pi}{\kappa}M = M - TS \end{aligned}$$

which implies

$$\frac{M}{2} = TS = \frac{\kappa}{8\pi}A \Longrightarrow S = \frac{A}{4},$$

from a tree-level calculation of Z.

3.3 Conformal anomalies and gauge theories on manifolds with boundary

If regularization breaks the invariance under conformal rescalings of the underlying classical theory, the resulting energy-momentum tensor acquires a non-vanishing trace, expressible in purely geometric terms via space-time methods, or through invariance theory and heat-equation methods.

3.3a Conformal anomaly in 4-D for massless spin-1/2 fields (local or spectral boundary conditions on the 4-ball).

3.3b Conformal anomaly for Euclidean Maxwell theory with boundary conditions on the potential and ghost fields.

3.3c Mixed boundary conditions in Euclidean quantum gravity at 1-loop level. Self-adjointness in the fully diffeomorphism-invariant case.

3.3d Simple supergravity on manifolds with boundary (lack of 1-loop finiteness already at this stage).

3.3e New invariants in the 1-loop divergences on manifolds with boundary, when tangential derivatives occur in the boundary operator.

3.3f Symbol techniques for the ellipticity of gauge theories on manifolds with boundary (their boundary-value problem at 1 loop).

3.3g Lack of strong ellipticity in Euclidean quantum gravity at 1 loop in the fully diffeomorphism-invariant case.

3.4 Euclidean version of the AdS/CFT correspondence

Basic idea: large N limits of certain conformal field theories in p dimensions can be described in terms of supergravity on the product of $(p+1)$-dimensional AdS space with a compact manifold. Thus, correlation functions in conformal field theory are given by the dependence of the supergravity action on the asymptotic behaviour at infinity (Maldacena, Witten).

3.4.1 Geometrical Framework

$B_{p+1} : \sum_{i=0}^{p} y_i^2 < 1$ open unit ball in R^{p+1}

$\mathrm{AdS}_{p+1} : B_{p+1}$ with line element

$$\mathrm{d}s^2 = 4\Big(1 - |y|^2\Big)^{-2} \sum_{i=0}^{p} \mathrm{d}y_i^2.$$

Compactified B_{p+1}: $\overline{B}_{p+1}$=closed unit ball: $\sum_{i=0}^{p} y_i^2 \leq 1$.

$\partial\overline{B}_{p+1} = S^p$: $\sum_{i=0}^{p} y_i^2 = 1$. The p-sphere is the Euclidean version of conformally compactified Minkowski space-time. The metric on B_{p+1} is singular at $|y| = 1$, and to obtain a metric which can be extended on $\overline{B}_{p+1}$ one can take a function f on $\overline{B}_{p+1}$, positive on B_{p+1}, e.g. $f = 1 - |y|^2$. One then performs the conformal rescaling $\mathrm{d}\tilde{s}^2 = f^2 \mathrm{d}s^2$, where f has a first-order zero at $|y| = 1$. The conformally rescaled metric restricts to a metric on S^p, and the use of $f\mathrm{e}^w$ is equally good, with w real-valued on $\overline{B}_{p+1}$.

For the Euclidean AdS_{p+1}, the metric is invariant under $SO(1, p+1)$, while its S^p boundary has only conformal structure, preserved by the action of $SO(1, p+1)$.

3.4.2 AdS/CFT for Gravity

$Z_{CFT}(h)$=partition function of CFT formulated on a S^4 with conformal structure h

AdS/CFT: $Z_{CFT}(h) = Z_S(h)$=supergravity partition function, obtained by integrating over metrics with a double pole near the boundary and inducing, on the boundary, the given conformal structure h.

Classical SUGRA: one first finds a solution g of the field equations with pre-assigned boundary behaviour, and then one takes $Z_S(h) = \mathrm{e}^{-I_S(g)}$.

In other words, given the supergravity partition function Z_S, which is a functional of boundary values of massless fields, this is *interpreted* as the generating functional of CFT correlation functions, for operators whose sources are the assigned boundary values.

3.4.3 A Simpler Example

Let us consider a massive non-minimally coupled scalar field on AdS with action functional

$$I = -\frac{1}{2} \int \mathrm{d}^{p+1}x\sqrt{g}\left[g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + (m^2 + \xi R)\phi^2\right].$$

This engenders two sets of modes:

$$\text{Regular modes}: \ \phi_R \sim L^{\triangle_+(\xi)},$$

$$\text{Irregular modes}: \ \phi_I \sim L^{\triangle_-(\xi)},$$

where L is a measure of the distance to the boundary (taken to be small), and

$$\triangle_\pm(\xi) \equiv \frac{p}{2} \pm \nu(\xi),$$

having defined

$$\nu(\xi) \equiv \sqrt{\frac{p^2}{4} + m^2 + \xi R}.$$

The irregular mode ϕ_I is normalizable only if

$$0 \leq \nu(\xi) < 1.$$

Moreover, the energy for ϕ_I is conserved, positive, finite only when

$$\xi = \frac{1}{2} \frac{\triangle_-(\xi)}{[1 + 2 \triangle_- (\xi)]}.$$

Although there are two possible quantizations in the bulk, the AdS/CFT prescription in the (original) form:

$$Z_{AdS}[\phi_0] = Z_{CFT}[\phi_0] = \left\langle \exp\left(\int_{\partial\Omega} \mathrm{d}^p x \ \mathcal{O}\phi_0 \right) \right\rangle,$$

only reproduces the conformal dimension $\triangle_+(\xi)$.

How to account for the missing conformal dimension $\triangle_-(\xi)$? (Klebanov and Witten 1999, Minces and Rivelles 2000, 2001).

4. QUANTUM COSMOLOGY

In the eighties, there has been a "renaissance" of quantum cosmology thanks to the Hartle–Hawking and Vilenkin proposals. According to the former, the quantum state of the universe is a functional integral over all Riemannian metrics on *compact* 4-manifolds having the Riemannian geometry (Σ, h) as their only boundary (the initial 3-surface shrinks to a point, hence the name "no-boundary proposal"; see Figures). Although this approach has deep roots in Riemannian geometry and quantum field theory, it is only a proposal, not derivable from first principles.

4.1 From ellipticity to surface states in quantum cosmology?

However, as far as 1-loop theory is concerned, boundary conditions on metric perturbations can be entirely derived from first principles (i.e. BRST rules for the functional integral and full invariance under infinitesimal diffeomorphisms on metric perturbations). Regrettably, for pure gravity in the de Donder gauge, this leads to lack of strong ellipticity if one wants to achieve an operator of Laplace type on metric perturbations.

A possible way out is provided by the use of non-local boundary conditions on metric perturbations. Instead of giving technical details, let us describe some key features of non-local boundary conditions in a simpler example, motivated by the Schröder analysis of Laplace operators and Bose–Einstein condensation.

Let us consider a function $q \in L_1(\mathbf{R}) \cap L_2(\mathbf{R})$, with the associated

$$q_R(x) \equiv \frac{1}{2\pi R} \sum_{l=-\infty}^{\infty} \mathrm{e}^{ilx/R} \int_{-\infty}^{\infty} \mathrm{e}^{-ily/R} q(y) \mathrm{d}y.$$

The function q_R is, by construction, periodic with period $2\pi R$, and tends to q as R tends to ∞. On considering the region

$$B_R \equiv \left\{ x, y : x^2 + y^2 \leq R^2 \right\},$$

one studies the Laplacian acting on square-integrable functions on B_R, with non-local boundary conditions given by

$$[u_{;N}]_{\partial B_R} + \oint_{\partial B_R} q_R(s - s') u(R\cos(s'/R), R\sin(s'/R)) \mathrm{d}s' = 0.$$

In polar coordinates, the resulting boundary-value problem reads as

$$-\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \right) u = Eu,$$

$$\frac{\partial u}{\partial r}(R,\varphi) + R\int_{-\pi}^{\pi} q_R(R(\varphi-\theta))\; u(R,\theta)\mathrm{d}\theta = 0.$$

For example, when the eigenvalue E is positive, the corresponding eigenfunction is

$$u_{l,E}(r,\varphi) = J_l(r\sqrt{E})\mathrm{e}^{\mathrm{i}l\varphi}.$$

On denoting by $\widetilde{q}$ the Fourier transform of q one finds therefore, by virtue of the boundary conditions, an equation leading implicitly to the evaluation of positive eigenvalues, i.e.

$$\left[\sqrt{E}J_l'(R\sqrt{E}) + J_l(R\sqrt{E})\widetilde{q}(l/R)\right] = 0.$$

The solutions which decay exponentially with increasing distance from the boundary (at least if this distance is $<< R$) are the *surface states*, whereas the solutions which remain non-negligible are called bulk states.

Our programme: given the Laplacian acting on symmetric rank-2 tensor fields on a portion of E^n bounded by S^{n-1}, one can build the associated non-local boundary-value problem leading to surface states. In Euclidean quantum gravity at 1 loop, this engenders from first principles (i.e. ellipticity and gauge invariance) a universe which starts in a quantum state and then becomes classical (such a transition is otherwise unclear).

Problem: this cannot hold *verbatim*, since a non-local boundary operator for gravity leads to a pseudo-differential operator on metric perturbations.

4.2 Variable cosmological constant in the RG-approach

Main idea of the Renormalization Group (RG) approach: to "integrate out" all fluctuations with momenta larger than some cutoff k, and to take them into account by means of a modified dynamics for the remaining fluctuation modes with momenta smaller than k. This modified dynamics is governed by a scale-dependent effective action Γ_k, the k-dependence of which is described by a functional differential equation, the exact Renormalization Group equation.

Flow equations can also be used for a complete quantization of fundamental theories. If the latter has the classical action I one imposes the initial condition $\Gamma_\kappa = I$ at the ultraviolet (UV) cut-off scale κ, uses the exact RG equation to compute Γ_k for all $k < \kappa$, and then sends $k \to 0$ and $\kappa \to \infty$. The defining property of a fundamental theory is that the "continuum limit" $\kappa \to \infty$ actually exists after the renormalization of finitely many parameters in the action; only a finite number of generalized couplings in Γ_0 is un-determined and has to be found on observational ground. This occurs for perturbatively renormalizable theories but, interestingly, there are also perturbatively non-renormalizable theories which admit a $\kappa \to \infty$ limit. The "continuum" limit of these non-perturbatively renormalizable theories is taken at a non-Gaussian fixed point of the RG flow. It replaces the Gaussian fixed point which, at least implicitly, underlies the construction of perturbatively renormalizable theories. Knowledge of the fixed-point structure is therefore crucial if one wants to understand whether a given model is really a fundamental theory.

Recently, Lauscher and Reuter have constructed a new exact RG equation for the effective average action of Euclidean quantum gravity. They have found both a Gaussian and a non-Gaussian fixed point. *If* the non-Gaussian fixed point occurs in the exact theory, quantum general relativity is likely to be renormalizable at non-perturbative level. A strong evidence has been found in favour of 4-dimensional Einstein gravity being asymptotically safe (i.e. non-perturbatively renormalizable).

Even more recently, Bonanno, myself and Rubano have applied the Arnowitt–Deser–Misner (ADM) formalism and the Dirac–Bergmann theory of constrained Hamiltonian systems to such a class of gravitational models, where both G and Λ are variable. A modified action functional has been built which reduces to the Einstein–Hilbert action when G is constant, and leads to a power-law growth of the scale factor for pure gravity and for a massless ϕ^4 theory in a Universe with Robertson–Walker symmetry, in agreement with the recently developed fixed-point cosmology, where

$$G(t) = g_\star \xi^{-2} t^2, \;\; \Lambda(t) = \lambda_\star \xi^2 t^{-2}.$$

Interestingly, *the renormalization-group flow at the fixed point is found to be compatible with a Lagrangian description of the running parameters* G *and* Λ.

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