An Enhanced CMB Power Spectrum from Quantum Gravity

D. Bini, G. Esposito, C. Kiefer, M. Krämer, F. Pessina

Quantum Theory of Gravity

The fundamental interaction that has not been quantized as yet is **Gravitation**

A deeper understanding of the quantum version

$\Downarrow$

Find a unified theory

The real structure of nature

Quantum Theory of Gravity

There exist many approaches to a quantum theory of gravity

nowadays no less than 16 !

Either field-theoretical or of sharply different nature.
Characteristic scale of the theory: **Planck scale**

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-33}\text{cm},$$

$$t_P = \frac{l_P}{c} = \sqrt{\frac{\hbar G}{c^5}} \approx 5.40 \times 10^{-44}\text{s},$$

$$m_P = \frac{\hbar}{l_P c} = \sqrt{\frac{\hbar c}{G}} \approx 1.22 \times 10^{19}\text{GeV}$$

How can we find a way?

Very difficult to test the effects in laboratory

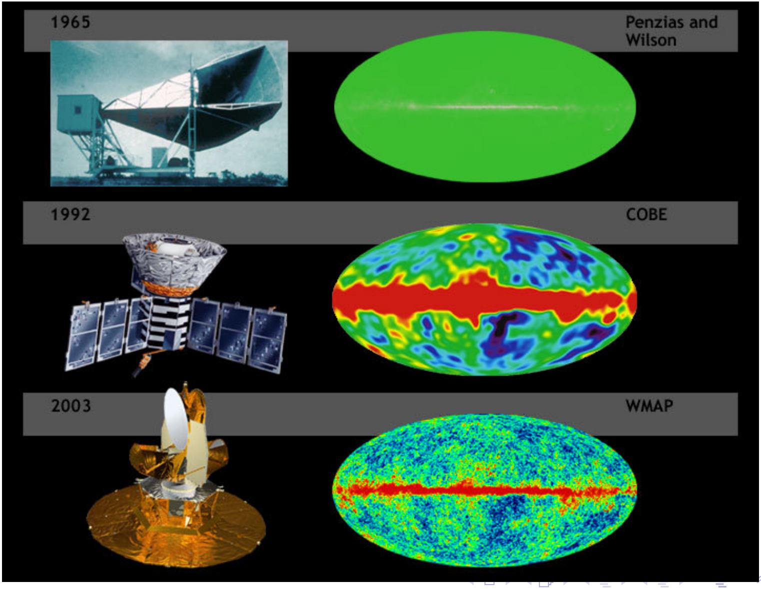
$\Downarrow$

Possible relevant effects at cosmological scale

$\Downarrow$

Cosmic Microwave Background Radiation (CMB)

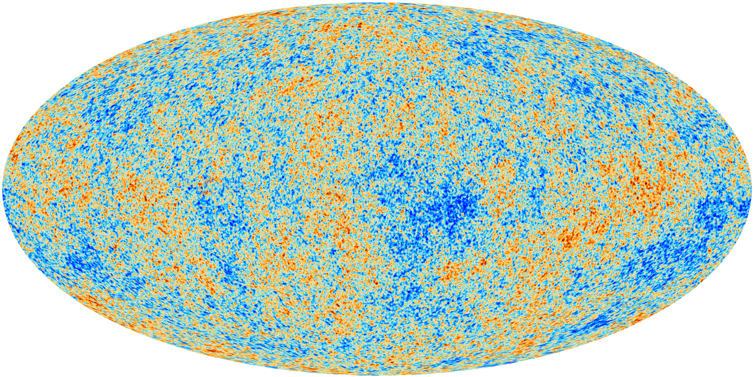
CMB measurement history



CMB measurement history

D. Bini,
G. Esposito,
C. Kiefer,
M. Krämer,
F. Pessina

Planck 2013



Einstein–Hilbert action

$$S = \frac{1}{16\pi} \int_M \mathrm{d}^4x \sqrt{-g}\,^{(4)}R$$

$$\Downarrow$$

Arnowitt–Deser–Misner formalism

$$\sqrt{-g}\,^{(4)}R = N\sqrt{h}\left(^{(3)}R + K_{ij}K^{ij} - K^2\right) + 2\partial_t(\sqrt{h}K)$$
$$-2\partial_i\left[\sqrt{h}(N^iK - g^{ij\,(3)}\nabla_j N)\right]$$

$$K_{ij} = \frac{1}{2N}\left(^{(3)}\nabla_i N_j + {}^{(3)}\nabla_j N_i - \frac{\partial h_{ij}}{\partial t}\right)$$

Hamiltonian formalism

$$H = \int \mathrm{d}^3x \left(N\mathscr{H}_t + N_i \chi^i \right)$$

This formalism enables us to use the Dirac quantization method

$$\hat{h}_{ij}\psi = h_{ij}\psi \qquad \hat{\pi}^{jk}\psi = \frac{\hbar}{\mathrm{i}} \frac{\delta\psi}{\delta h_{jk}}$$

$$\Downarrow$$

Wheeler–DeWitt equation (WDW)

$$\hat{\mathscr{H}}_t\psi = \left\{ -\hbar^2 G_{ijkl} \frac{\delta^2}{\delta h_{ij} \delta h_{kl}} - \sqrt{h}^{(3)}R \right\} \psi[h_{ij}] = 0$$

Friedmann–Lemaitre–Robertson–Walker (FLRW) Universe

D. Bini, G. Esposito, C. Kiefer, M. Krämer, F. Pessina

Considering a **spatially flat**, **homogeneus** and **isotropic** Universe, one can describe it by a FLRW metric

$$ds^2 = d\tau^2 - a^2(\tau)\delta_{ij}dx^i dx^j.$$

The Wheeler–DeWitt equation, if one assumes an **inflationary field** ϕ, becomes

$$\left[\frac{1}{m_P^2}\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2} + e^{6\alpha}m^2\phi^2\right]\psi(\alpha,\phi) = 0 \quad \alpha = \ln a$$

Slow-Roll Condition

We can get a simpler form of the WDW equation if we assume the **slow-roll condition** for the inflationary field ϕ

$$\frac{\partial^2 \psi}{\partial \phi^2} \ll e^{6\alpha} m^2 \phi^2 \psi \quad m\phi \rightarrow m_P H$$

Thus the equation becomes

$$\left[\frac{1}{m_P^2} \frac{\partial^2}{\partial \alpha^2} + e^{6\alpha} m_P^2 H^2 \right] \psi(\alpha, \phi) = 0$$

Born–Oppenheimer Approximation and Inhomogeneous Fluctuations

We now consider the fluctuations of an inhomogeneous inflaton field on top of its homogeneous part

$$\phi \to \phi(t) + \delta\phi(\mathbf{x}, t) \quad \delta\phi(\mathbf{x}, t) = \sum_{\kappa} f_{\kappa}(t) e^{i\kappa\cdot x}$$

The smallness of the fluctuations' self-interaction and the **Born–Oppenheimer (BO)** approximation enable us to factorize the wave functional

$$\psi(\alpha, \phi, \{f_\kappa\}_{\kappa=1}^{\infty}) = \psi_0(\alpha, \phi) \prod_{\kappa=1}^{\infty} \tilde{\psi}_\kappa(\alpha, \phi, f_\kappa)$$

Hamiltonian Factorization

Thus the WDW equation can be rewritten

$$\left[\mathcal{H}_0 + \sum_{\kappa=1}^{\infty} \mathcal{H}_\kappa\right] \psi(\alpha, \phi, \{f_\kappa\}_{\kappa=1}^{\infty}) = 0$$

$$\mathcal{H}_0 = \frac{e^{-3\alpha}}{2}\left[\frac{1}{m_P^2}\frac{\partial^2}{\partial\alpha^2} + e^{6\alpha} m_P^2 H^2\right]$$

$$\mathcal{H}_\kappa = \frac{e^{-3\alpha}}{2}\left[-\frac{\partial^2}{\partial f_\kappa^2} + W_\kappa(\alpha) f_\kappa^2\right]$$

$$W_\kappa(\alpha) = \kappa^2 e^{4\alpha} + m^2 e^{6\alpha}$$

The Jeffreys–Wentzel–Kramers–Brillouin Method

Identify the quantum gravitational contributions to the terms of the expansion of the WDW in powers of m_P^2
(Effective Theory)

On writing every single mode in the form

$$\psi_\kappa(\alpha, f_\kappa) = e^{iS(\alpha, f_\kappa)} \quad S(\alpha, f_\kappa) = m_P^2 S_0 + m_P^0 S_1 + m_P^{-2} S_2 + \ldots$$

m_P^0 Order

We note that at the zeroth stage of the JWKB approximation one obtains the usual evolution equation for matter **Schwinger–Tomonaga**

$$i\frac{\partial}{\partial t}\psi_\kappa^{(0)} = \mathcal{H}_\kappa\psi_\kappa^{(0)} \quad \psi_\kappa^{(0)} \equiv \gamma(\alpha)e^{iS_1(\alpha,f_\kappa)}$$

Where we have defined the **JWKB time**

$$\frac{\partial}{\partial t} \equiv -e^{-3\alpha}\frac{\partial S_0}{\partial\alpha}\frac{\partial}{\partial\alpha}$$

m_P^2 Order

To second order we obtain the first **quantum-gravitational corrections** to the matter wave functional

$$\mathrm{i}\frac{\partial \psi_\kappa^{(1)}}{\partial t} = \mathcal{H}_\kappa \psi_\kappa^{(1)} - \frac{e^{3\alpha}}{2m_P^2 \psi_\kappa^{(0)}} \left[\frac{(\mathcal{H}_\kappa)^2}{V(\alpha)} \psi_\kappa^{(0)} + \mathrm{i} \left(\frac{\psi_\kappa^{(0)}}{V(\alpha)} \frac{\partial \mathcal{H}_\kappa}{\partial t} - \frac{1}{V^2(\alpha)} \frac{\partial V(\alpha)}{\partial t} \mathcal{H}_\kappa \psi_\kappa^{(0)} \right) \right] \psi_\kappa^{(1)}$$

$$\psi_\kappa^{(1)}(\alpha, f_\kappa) \equiv \psi_\kappa^{(0)}(\alpha, f_\kappa) e^{\mathrm{i}\frac{\eta_2(\alpha, f_\kappa)}{m_P^2}}$$

Gaussian Hypothesis

By making a Gaussian ansatz

$$\psi_{\kappa}^{(0)}(t, f_{\kappa}) = \mathcal{N}_{\kappa}^{(0)} \mathrm{e}^{-\frac{1}{2}\Omega_{\kappa}^{(0)} f_{\kappa}^{2}}$$

we obtain a coupled system of non-linear differential equations

$$\dot{\mathcal{N}}_{\kappa}^{(0)}(t) = -\mathrm{i}\frac{\mathrm{e}^{-3\alpha}}{2}\mathcal{N}_{\kappa}^{(0)}(t)\Omega_{\kappa}^{(0)}(t)$$

$$\dot{\Omega}_{\kappa}^{(0)}(t) = \mathrm{i}\mathrm{e}^{-3\alpha}\left[-(\Omega_{\kappa}^{(0)}(t))^{2} + W_{\kappa}(t)\right]$$

$\Omega_{\kappa}^{(0)}$ Solution

On defining

$$\xi = \frac{\kappa}{Ha(t)} \quad \mu = \frac{m}{H} \quad \nu = \frac{1}{2}\sqrt{9 - 4\mu^2} \quad h = \frac{H^2}{\kappa^3}$$

we get the solution

$$\Omega_{\kappa}^{(0)}(\xi) = \frac{1}{h\xi^2} \frac{1}{(C_1 Y_\nu(\xi) + J_\nu(\xi))}$$

$$\times \left[-\mathrm{i} C_1 Y_{\nu+1}(\xi) + \frac{\mathrm{i}}{2\xi} \left(C_1 Y_\nu(\xi)(3+2\nu) - 2\xi J_{\nu+1}(\xi) + J_\nu(\xi)(3+2\nu) \right) \right]$$

Massless Case

We find that the solution of the equation considering $\mu \ll 1$ coincides, in the limit $\mu \to 0$, i.e. $\nu \to \frac{3}{2}$, with the solution of the complete equation

$$\Omega_{\kappa}^{(0)}(\xi) = \frac{\mathrm{i}}{h\xi} \frac{C_1 \cos\xi - \sin\xi}{[(C_1 + \xi)\cos\xi + (C_1\xi - 1)\sin\xi]}$$

$$= \frac{\mathrm{i}}{h\xi} \frac{\left(C_1 J_{-\frac{1}{2}} - J_{\frac{1}{2}}\right)}{\left[(C_1 + \xi) J_{-\frac{1}{2}} + (C_1\xi - 1) J_{\frac{1}{2}}\right]}$$

This solution can be re-expressed substuting $C_1 = \zeta e^{i\beta}$ so that

$$\Omega_k^{(0)}(\xi) = \frac{k^3}{H^2} \frac{i}{\xi} \frac{AB^*}{|B|^2}$$

where

$$A = \rho + i\sigma \quad B = \gamma + i\delta$$

$$\rho = 2(\zeta \cos \beta \cos \xi - \sin \xi) \qquad \sigma = 2\zeta \sin \beta \cos \xi$$

$$\gamma = 2\zeta \left[\cos \beta(\cos \xi + \xi \sin \xi) - (\sin \xi - \xi \cos \xi)\right]$$

$$\delta = 2\zeta \sin \beta[\cos \xi + \sin \xi]$$

Power Spectrum

We define the **power spectrum**

$$\mathcal{P}^{(0)}(k) := \frac{k^3}{2\pi^2} \left|\delta_k(t_{\rm enter})\right|^2$$

where

$$\delta_k(t_{\rm enter}) = \frac{4}{3} \frac{\dot{\sigma}_k(t)}{\dot{\phi}(t)}\Big|_{t=t_{\rm exit}}$$

and we have set

$$\sigma_\kappa^2(t) \equiv \langle \psi_\kappa | f_\kappa^2 | \psi_\kappa \rangle = \frac{1}{2\Re \mathrm{e}\Omega_\kappa(t)}$$

Power Spectrum

In light of the definition of σ_κ one has

$$|\dot{\sigma}_\kappa(t)| = \left| \frac{H\xi}{\sqrt{2}} \frac{\mathrm{d}}{\mathrm{d}\xi} \left[\left(\Re\mathrm{e}\Omega_\kappa(\xi) \right)^{-\frac{1}{2}} \right] \right|$$

At m_P^0 order we have for the general solution $\Omega_\kappa^{(0)}$

$$\left| \dot{\sigma}_k^{(0)}(t) \right|_{t_{\rm exit}} = \frac{2\sqrt{2}\pi^2 H^2}{k^{\frac{3}{2}}} \left| \frac{\sqrt{\zeta}(\zeta + 2\pi \cos\beta)}{\sqrt{\sin\beta}\sqrt{\zeta^2 + 4\pi\cos\beta + 4\pi^2}} \right|$$

Power Spectrum

At m_P^0 order if we consider the **(Bunch–Davies Vacuum)** boundary condition

$$\Omega_\kappa^{(0)}(\infty) = \frac{1}{h\xi^2}$$

we obtain at the $\xi(t_{exit}) = 2\pi$ time

$$\left|\dot{\sigma}_\kappa^{(0)}\right| = \frac{H^2}{\kappa^{\frac{3}{2}}} \frac{2\sqrt{2}\pi^2}{\sqrt{4\pi^2+1}}$$

Power Spectrum: Quantum Corrections

At m_P^2 order, making the same Gaussian ansatz, we can write the wave functional in the form

$$\psi_\kappa^{(1)}(t, f_\kappa) = \left(\mathcal{N}_\kappa^{(0)}(t) + \frac{1}{m_P^2}\mathcal{N}_\kappa^{(1)}(t)\right) \exp\left[-\frac{1}{2}\left(\Omega_\kappa^{(0)}(t) + \frac{1}{m_P^2}\Omega_\kappa^{(1)}(t)\right) f_\kappa^2\right]$$

and inserting it into the m_P^2 order equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\log\left(N_k^{(0)} + \frac{N_1^{(1)}}{m_P^2}\right) - \frac{\mathrm{i}}{2}\left(\dot{\Omega}_k^{(0)} + \frac{\dot{\Omega}_k^{(1)}}{m_P^2}\right) f_k^2 =$$

$$\frac{1}{2}\mathrm{e}^{-3\alpha}\left\{\Omega_k^{(0)} + \frac{1}{m_P^2}\left[\Omega_k^{(1)} - \frac{3}{4V}\left(\left(\Omega_k^{(0)}\right)^2 - \frac{2}{3}W_k\right)\right] +\right.$$

$$\left.\left[W_k - \left(\Omega_k^{(0)} + \frac{\Omega_k^{(1)}}{m_P^2}\right)^2 - \frac{3\Omega_k^{(0)}(W_k - (\Omega_k^{(0)})^2)}{2Vm_P^2}\right] f_k^2 + O(f_k^4)\right\}$$

Power Spectrum: Quantum Corrections

The equation for $\Omega_k^{(1)}$ is

$$\dot{\Omega}_\kappa^{(1)}(t) = -2\mathrm{i}e^{-3\alpha}\Omega_\kappa^{(0)}(t)\left[\Omega_\kappa^{(1)}(t) - \frac{3}{4V(t)}\left((\Omega_\kappa^{(0)}(t))^2 - W_\kappa\right)\right]$$

if we substitute the massless expression for $\Omega_k^{(0)}$ with the Bunch-Davies boundary condition

$$\frac{\mathrm{d}\Omega_k^{(1)}}{\mathrm{d}\xi} = \frac{2\mathrm{i}\xi}{(\xi-\mathrm{i})}\Omega_k^{(1)} + \frac{3}{2}\xi^3\frac{(2\xi-\mathrm{i})}{(\xi-\mathrm{i})^3}$$

Power Spectrum: Quantum Corrections

One can factorize the correction contributions

$$\left|\dot{\sigma}_\kappa^{(1)}(t)\right| = \left|\sigma_\kappa^{(0)}\right| |C_\kappa|$$

$$C_k(\xi) \equiv \left(1 + \frac{\xi^2+1}{\kappa^3}\frac{H^2}{m_P^2}\Re\mathrm{e}\,\Omega_\kappa^{(1)}(\xi)\right)^{-\frac{3}{2}} \left(1 - \frac{(\xi^2+1)^2}{2\xi\kappa^3}\Re\mathrm{e}\left[\frac{\mathrm{d}}{\mathrm{d}\xi}\Omega_\kappa^{(1)}(\xi)\right]\frac{H^2}{m_P^2}\right)$$

In order to evalute this quantity we have to find the function $\Omega_\kappa^{(1)}$, that is, for the **massless form** of $\Omega_\kappa^{(0)}$, and considering the boundary condition $\Omega_\kappa^{(1)}(0) = 0$

$$\Omega_\kappa^{(1)}(\xi) = \frac{-3e^{2i\xi}}{8}\frac{1+\mathrm{Ei}(1,2)e^2}{(1+i\xi)^2} + \frac{3}{8}\frac{1+6i\xi+4\mathrm{Ei}(1,2i\xi+2)e^{2i\xi+2}-4\xi^2-4i\xi^3}{(1+i\xi)^2}$$

$$\mathrm{Ei}(a,z) \equiv \int_1^\infty \frac{e^{-tz}}{t^a}\mathrm{d}t \quad a \in \mathbb{R} \text{ and } \Re\mathrm{e}(z) > 0$$

Power Spectrum: Quantum Corrections

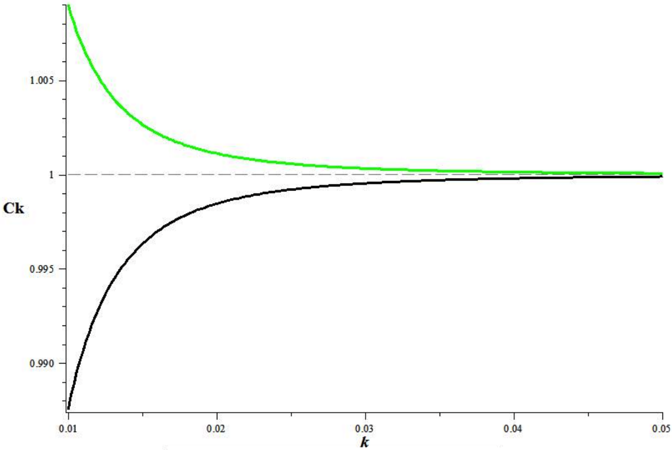
At $t_{exit} \to \xi = 2\pi$ time

$$C_k \equiv \left(1 - \frac{54.37}{\kappa^3}\frac{H^2}{m_P^2}\right)^{-\frac{3}{2}} \left(1 + \frac{7.98}{\kappa^3}\frac{H^2}{m_P^2}\right)$$

and for what concerns the power spectrum

$$\mathcal{P}^{(1)}(k) = \mathcal{P}^{(0)}(k)\, C_k^2 \sim \mathcal{P}^{(0)}(k)\left[1 + \frac{89.54}{k^3}\frac{H^2}{m_P^2} + \frac{1}{k^6}\mathrm{O}\left(\frac{H^4}{m_P^4}\right)\right]^2$$

The C_κ Behavior at Various Scales



Equation in the z Variable

Remarkably, by passing to the new variable

$$z = 1 + \mathrm{i}\xi$$

the m_P^2 order equation can be written

$$\frac{\mathrm{d}\Omega_k^{(1)}}{\mathrm{d}z} = 2\left(1 - \frac{1}{z}\right)\Omega_k^{(1)} + \frac{3}{2}\left(7 - 2z - \frac{9}{z} + \frac{5}{z^2} - \frac{1}{z^3}\right)$$

that leads to the solution

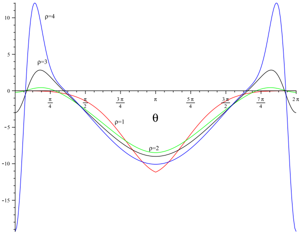
$$\Omega_k^{(1)}(z) = P_1 \frac{\mathrm{e}^{2z}}{z^2} + \frac{3}{8z^2}\left[4z^3 - 8z^2 + 10z - 5 + 4\mathrm{e}^{2z}\mathrm{Ei}(1, 2z)\right]$$

Graphical studies

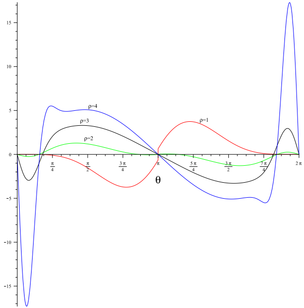
Such a solution can be studied graphically by introducing the complex polar representation for $z = \rho e^{i\theta}$ and defining the functions

$$f_\rho(\theta) = \mathrm{Re}\left[\Omega_k^{(1)}(\rho e^{i\theta})\right] \quad g_\rho(\theta) = \mathrm{Im}\left[\Omega_k^{(1)}(\rho e^{i\theta})\right]$$

$f_\rho(\theta)$



$g_\rho(\theta)$



Observability of the Corrections

We note that the uncorrected power spetrum is proportional to

$$\mathcal{P}^{(0)}(k) \propto \frac{H^4}{\left|\dot{\phi}(t)\right|^2_{t_{\rm exit}}}$$

this corresponds, apart from a dimensionless constant, to the standard power spectrum of scalar cosmological perturbations

$$\mathcal{P}_s^{(0)}(k) = \frac{G}{\epsilon\pi} H^2$$

where we have introduced the first slow-roll parameter

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{4\pi G \left|\dot{\phi}\right|^2_{t_{\rm exit}}}{H^2}$$

Quantum Corrections

The quantum correction takes the approximate form

$$C_k^2 = 1 + \delta^{\pm}_{\mathrm{WDW}}(k) + \frac{1}{k^6}\,\mathcal{O}\left(\left(\frac{H}{m_{\mathrm{P}}}\right)^4\right)$$

where $\delta^{\pm}_{\mathrm{WDW}}(k)$ could take the values

$$\delta^{+}_{\mathrm{WDW}}(k) = \frac{179.09}{k^3}\left(\frac{H}{m_{\mathrm{P}}}\right)^2 \qquad \delta^{-}_{\mathrm{WDW}}(k) = -\frac{247.68}{k^3}\left(\frac{H}{m_{\mathrm{P}}}\right)^2$$

Spectral Index

The basic equations in the theory of the spectral index n_s and its running α_s are

$$n_s - 1 := \frac{\mathrm{d}\log \mathcal{P}_s}{\mathrm{d}\log k} \approx \frac{1}{H}\frac{\mathrm{d}\log \mathcal{P}_s}{\mathrm{d}t} \approx 2\eta - 4\epsilon - 3\delta^{\pm}_{\mathrm{WDW}}$$

$$\alpha_s := \frac{\mathrm{d}n_s}{\mathrm{d}\log k} \approx 2(5\epsilon\eta - 4\epsilon^2 - \Xi^2) + 9\delta^{\pm}_{\mathrm{WDW}}$$

where we have defined the slow-roll parameters

$$\eta := -\frac{\ddot{\phi}}{H\dot{\phi}} \quad \Xi^2 := \frac{1}{H^2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\ddot{\phi}}{\dot{\phi}}.$$

Observability bounds

Reinserting a reference wave number wich can either correspond

to $k_{min} \approx 1.4 \times 10^{-4}\ \mathrm{Mpc}^{-1}$ largest observable scale or to $k_0 = 0.002\ \mathrm{Mpc}^{-1}$ pivot scale

we find the corrections for $k \to \frac{k}{k_0}$

$$\left|\delta^{+}_{\mathrm{WDW}}(k_0)\right| \lesssim 2.9 \times 10^{-9}, \quad \left|\delta^{-}_{\mathrm{WDW}}(k_0)\right| \lesssim 4.0 \times 10^{-9}$$

and for $k \to \frac{k}{k_{min}}$

$$\left|\delta^{+}_{\mathrm{WDW}}(k_0)\right| \lesssim 9.8 \times 10^{-13}, \quad \left|\delta^{-}_{\mathrm{WDW}}(k_0)\right| \lesssim 1.4 \times 10^{-12}$$

the resulting upper bounds for H are

$$H \lesssim 1.67 \times 10^{-2}\, m_{\mathrm{P}} \approx 4.43 \times 10^{17}\ \mathrm{GeV}$$

$$H \lesssim 1.42 \times 10^{-2}\, m_{\mathrm{P}} \approx 3.76 \times 10^{17}\ \mathrm{GeV}$$

Conclusions

- ⋆ Exact form of the functions $\Omega^{(0)}_{\kappa}$, $\Omega^{(1)}_{\kappa}$ and $\dot{\sigma}^{(0)}_{\kappa}$. Enhancement/Suppression of quantum gravitational corrections, hard to discriminate on observational ground.
- ⋆ Unobservable corrections to CMB anisotropy spectrum; nevertheless, their size is bigger than QG corrections in laboratory situations.
- ⋆ Other choices of vacuum besides Bunch–Davies allowed by the general integral of our non linear equation?

Conclusions

- ⋆ Calculation of an upper limit for H in an inflationary model.
- ⋆ Gauge-invariant Mukhanov variables instead of a scalar field.
- ⋆ More complicated quantum state (instead of ground state) to see how the results depend on this choice.
- ⋆ We have found a way of dealing with unitarity violating terms.

Bibliography

- A. Yu. Kamenshchik, A. Tronconi, G. Venturi, Inflation and Quantum Gravity in a Born–Oppenheimer Context
- D. Bini, G. Esposito, C. Kiefer, M. Kramer, F. Pessina, Phys. Rev. D87, 104008 (2013)
- M. Krämer, arXiv:1303.1544 [gr-qc]
- G. Calcagni, arXiv:1209.0473v1 [gr-qc]
- C. Kiefer, M. Krämer, Int. J. Mod. Phys. D 21, 1241001 (2012)
- C. Kiefer, M. Krämer, Phys. Rev. Lett. 108, 021301 (2012)
- C. Bertoni, F. Finelli, G. Venturi, Class. Quantum Grav. 13, 2375 (1996)
- C. Kiefer, T. P. Singh, Phys. Rev. D44, 1067 (1991)