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2010 J. Phys.: Conf. Ser. 228 012052

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Relativistic orbits and Gravitational Waves from gravitomagnetic corrections

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Abstract. Corrections to the relativistic theory of orbits are discussed considering higher order approximations induced by gravitomagnetic effects. Beside the standard periastron effect of General Relativity (GR), a new nutation effect was found due to the c^{-3} orbital correction. According to the presence of that new nutation effect we studied, via the quadrupole approximation, the gravitational waveforms emitted by a compact object (neutron star (NS) or black hole (BH)) orbiting around a massive black hole (MBH). To obtain the emitted gravitational wave (GW) amplitudes, a numerical solution of the equations is given in different mass ratios and initial conditions. We conclude that the effects we studied could be of interest for the future space laser interferometric GW antenna LISA.

1. Introduction

The magnetic field is produced by the motion of electric-charge, i.e. the electric current. The analogy with gravity consists in the fact that a mass-current can produce a "gravitomagnetic" (GM) field. The formal analogy between electromagnetic and gravitational fields was explored in the framework of GR [1], it was also shown that a rotating mass generates a gravitomagnetic field. We want to study how the relativistic theory of orbits and the production of GW is affected by GM corrections. The corrections, which are off-diagonal terms in the metric, can be seen as further powers in the expansion in c^{-1} (up to c^{-3}). Nevertheless, the effects on the orbit behaviour involves not only the precession at peri-astron but also nutation corrections. Our approach suggests that, in the weak field approximation, when considering higher order corrections in the equations of motion, the GM effects can be particularly significant. In systems approaching to strong field regimes, these corrections give rise to chaotic behaviors in the transients dividing stable from unstable orbits [2]. In general, such contributions are discarded since they are assumed to be too small but they have to be taken into account as soon as the v/c ratio is significant. It is important to stress that we are not investigating simply frame-dragging effects in the Post-Newtonian (PN) approach [3] but the GM effects emerging from the weak-field approximation. In this sense, as soon as higher order terms in v/c come out, their effect could give rise to, in principle, observable signatures both in orbital and in GW emission (see [2]).

2. Gravitomagnetic effects: the weak field limit

In the framework of the linearized weak-field and slow motion approximation of GR, the ensemble of the so-called GM effects are induced by the off-diagonal components of the space-time metric tensor that are proportional to the components of the matter current density of the source. It is possible to take into account two types of mass currents in gravity. The former is induced by the matter source rotation around its center of mass: it generates the intrinsic GM field which is closely related to the angular momentum (spin) of the rotating body. The latter is due to the translational motion of the source: it is responsible of the extrinsic GM field. Starting from the Einstein field equations in the weak field approximation, one obtains the GM equations and then the corrections in the metric [4].

Starting from the weak field approximation of the gravitational field:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \quad |h_{\mu\nu}(x)| \ll 1$$

where $\eta_{\mu\nu}$ is the Minkowski tensor and $h_{\mu\nu}$ a small perturbation¹, in the approximation $|\mathbf{v}| \ll c$ and by using the Einstein equations and the harmonic gauge condition, one finds that, up to leading order in v/c , the metric is determined by the usual gravitational potential Φ and by a vector potential V^l ,

$$V^l = -G \int \frac{\rho v^l}{|\mathbf{x} - \mathbf{x}'|} d^3x' . \quad (1)$$

depending on the orbital velocities v^l of the bodies. The metric results to be:

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \frac{8\delta_{lj}V^l}{c^3} c dt dx^j - \left(1 - \frac{2\Phi}{c^2}\right) \delta_{lj} dx^l dx^j \quad (2)$$

The difference with respect to the standard PN approximation is that the time-spatial terms are generally discarded, while here, as we will see, they give rise to observable effects. This fact could be immediately seen on the geodesic motion. By calculating the affine connection related to the metric (2), the geodesic equations are obtained:

$$\ddot{x}^\alpha + \Gamma_{\mu\nu}^\alpha \dot{x}^\mu \dot{x}^\nu = 0 \quad (3)$$

where the dot indicate differentiation with respect to the affine parameter. In order to put in evidence the GM contributions, let us explicitly calculate the Christoffel symbols at lower orders. By some straightforward calculations, one gets

$$\begin{aligned} \Gamma_{00}^0 &= 0 & \Gamma_{0j}^0 &= \frac{1}{c^2} \frac{\partial\Phi}{\partial x^j} & \Gamma_{ij}^0 &= -\frac{2}{c^3} \left(\frac{\partial V^i}{\partial x^j} + \frac{\partial V^j}{\partial x^i} \right) \\ \Gamma_{00}^k &= \frac{1}{c^2} \frac{\partial\Phi}{\partial x^k} & \Gamma_{0j}^k &= \frac{2}{c^3} \left(\frac{\partial V^k}{\partial x^j} - \frac{\partial V^j}{\partial x^k} \right) & \Gamma_{ij}^k &= -\frac{1}{c^2} \left(\frac{\partial\Phi}{\partial x^j} \delta_i^k + \frac{\partial\Phi}{\partial x^i} \delta_j^k - \frac{\partial\Phi}{\partial x^k} \delta_{ij} \right) \end{aligned} \quad (4)$$

Let us stress that we are considering the GM effects on the orbital motion, so the Christoffel symbols result accordingly corrected. Usually, in the standard weak field approximation, vector potentials are not considered. In our approximation, we retain terms up to order Φ/c^2 and V^j/c^3 . Our aim is to show that, in several cases like in tight binary stars, it is not correct to discard higher order terms in v/c since physically interesting effects could come out.

Considering only the spatial components of the geodesic equations:

$$\begin{aligned} \frac{d^2 x^k}{d\sigma^2} + \frac{1}{c^2} \frac{\partial\Phi}{\partial x^k} \left(c \frac{dt}{d\sigma} \right)^2 + \frac{1}{c^2} \frac{\partial\Phi}{\partial x^k} \delta_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} - \frac{2}{c^2} \frac{\partial\Phi}{\partial x^l} \frac{dx^l}{d\sigma} \frac{dx^k}{d\sigma} + \\ + \frac{4}{c^3} \left(\frac{\partial V^k}{\partial x^j} - \delta_{jm} \frac{\partial V^m}{\partial x^k} \right) c \frac{dt}{d\sigma} \frac{dx^j}{d\sigma} = 0 \end{aligned} \quad (5)$$

¹ Greek indices running from 0 to 3 and latin from 1 to 3

we obtain the orbit equations. Calling $dl_{euclid} = \delta_{ij} dx^i dx^j$, and $e^k = \frac{dx^k}{dl_{euclid}}$ we have, in vector form,

$$\frac{de}{dl_{euclid}} = -\frac{2}{c^2} [\nabla\Phi - \mathbf{e}(\mathbf{e} \cdot \nabla\Phi)] + \frac{4}{c^3} [\mathbf{e} \wedge (\nabla \wedge \mathbf{V})] \quad (6)$$

The GM term is the second one in Eq.(6) and it is usually discarded since considered not relevant. This is not true if v/c is non negligible as in the cases of tight binary systems or point masses approaching to BH.

We can then write the Lagrangian and derive the orbital equations of motion by means of the Euler-Lagrange equations (see [2]):

$$\ddot{r} = \frac{1}{cr(rc^2 + 2GM)} \left\{ c(rc^2 + GM) (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) r^2 + \right. \\ \left. - 4GM\dot{t} [\cos \theta (\cos \phi + \sin \phi) - \sin \theta] \dot{\theta} + \sin \theta (\cos \phi - \sin \phi) \dot{\phi} \right\} r + cGM\dot{r}^2 - cGM\dot{t}^2 \quad (7)$$

$$\ddot{\phi} = -\frac{2}{r^2(rc^3 + 2GMc)} \left\{ c \cot \theta (rc^2 + 2GM) \dot{\theta} \dot{\phi} r^2 + \right. \\ \left. + \dot{r} [2GM \csc \theta (\sin \phi - \cos \phi) \dot{t} + cr(rc^2 + GM) \dot{\phi}] \right\} \quad (8)$$

$$\ddot{\theta} = \frac{1}{r^2(rc^3 + 2GMc)} \left\{ c \cos \theta r^2 (rc^2 + 2GM) \sin \theta \dot{\phi}^2 + \right. \\ \left. + \dot{r} [4GM(\cos \theta (\cos \phi + \sin \phi) - \sin \theta) \dot{t} - 2cr(rc^2 + GM) \dot{\theta}] \right\} \quad (9)$$

and, with the condition $\mathcal{L} = 1$ ($m = 1, c = 1$), find an expression for \dot{r} that is a constrain equation related to the energy.

Beside the standard periastron precession of GR, if $\dot{r} \neq 0$, even with initial conditions $\dot{\theta} = 0$ and $\theta = \pi/2$, a nutation effect is induced by gravitomagnetism and stability depends on it. Our aim is to study how GM effects modify the orbital shapes and what are the relevant parameters to determine the stability of the problem.

3. Numerical approach to the solution

The solution of the system of differential equations (ODE) of motion (7-9) presents some difficulties since the equations are stiff. We have found solutions by using a Stiffness Switching Method to provide an automatic mean of switching between a non-stiff and a stiff solver coupled with a more conventional explicit Runge-Kutta method for the non-stiff part of differential equations. Time series of both $\dot{r}(t)$ and $\ddot{r}(t)$ together with the phase portrait $r(t), \dot{r}(t)$ are shown assuming as initial angular precession and nutation velocities values that are integer fractions of the initial radial velocity. In Fig. 1 we show as example the one with: $\dot{\phi} = \frac{1}{10}\dot{r}$ and $\dot{\theta} = \frac{1}{10}\dot{\phi}$

3.1. Orbits with gravitomagnetic corrections

In this section we show some examples of orbits. In Fig. 2 we show the orbits with (red-line) and without GM correction (black-line). Finally in Fig. 3 the phase portrait with (red-line) and without (blue-line) GM orbital correction is shown. The example we are showing was obtained by solving the system with the parameters and initial conditions: $\mu \approx 1M_{\odot}$, $E = 0.95$, $\phi_0 = 0$, $\theta_0 = \frac{\pi}{2}$, $\dot{\theta}_0 = \frac{1}{10}\dot{\phi}_0$, $\dot{\phi}_0 = -\frac{1}{10}\dot{r}_0$ and $\dot{r}_0 = -\frac{1}{10}$ and $r_0 = 20\mu$.

4. Gravitational waves in the quadrupole approximation

Considering the orbital equations (see [2]), we know that direct signatures of gravitational radiation are its amplitude and waveform [4]. These observations give information on the nature

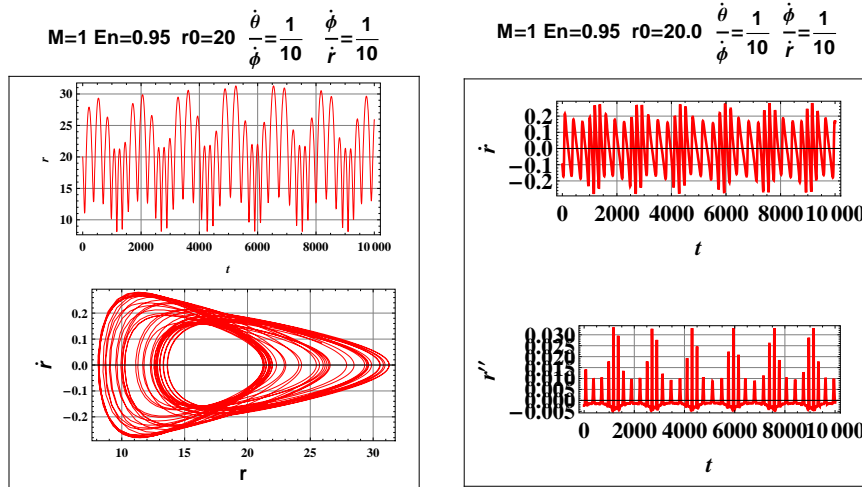


Figure 1. Plots of: $r(t)$ (upper left), phase portrait of $r(t)$ versus $\dot{r}(t)$ (bottom left), $\dot{r}(t)$ (upper right) and $\ddot{r}(t)$ (bottom right) system of reduced mass $1M_{\odot}$. The parameters and initial conditions are: $\mu \approx 1M_{\odot}$, $E = 0.95$, $\phi_0 = 0$, $\theta_0 = \frac{\pi}{2}$, $\dot{\theta}_0 = \frac{1}{10}\dot{\phi}_0$, $\dot{\phi}_0 = -\frac{1}{10}\dot{r}_0$ and $\dot{r}_0 = -\frac{1}{100}$ and $r_0 = 20\mu$. The stiffness is evident from the trend of $\dot{r}(t)$ and $\ddot{r}(t)$

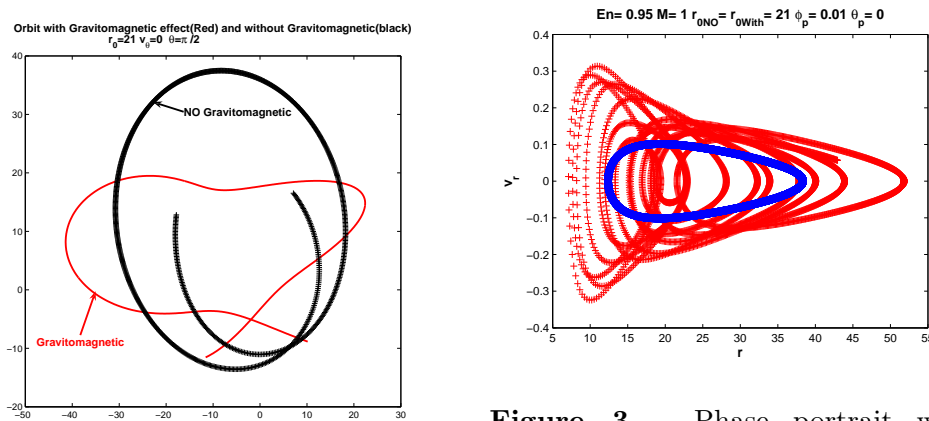


Figure 2. The orbit with GM correction (red-line) and without GM correction (black-line).

Figure 3. Phase portrait with (red-line) and without (blue-line) GM orbital correction for: $\mu \approx 1M_{\odot}$, $E = 0.95$, $\phi_0 = 0$, $\theta_0 = \pi/2$, $\dot{\theta}_0 = (1/10)\dot{\phi}_0$, $\dot{\phi}_0 = -(1/10)\dot{r}_0$ and $\dot{r}_0 = -1/10$ and $r_0 = 20\mu$.

of the GW source, on the propagating medium, and, in principle, on the gravitational theory producing such a radiation. The amplitude of GWs is, as usual, evaluated by

$$h^{jk}(t, R) = \frac{2G}{Rc^4} \ddot{Q}^{jk}, \quad (10)$$

being R the distance between the source and the observer, $\{j, k\} = 1, 2$, and Q_{ij} the quadrupole mass tensor

$$Q_{ij} = \sum_a m_a (3x_a^i x_a^j - \delta_{ij} r_a^2). \quad (11)$$

being G the Newton constant, r_a the modulus of the vector radius of the a -th particle and the sum running over all masses m_a in the system. We computed the amplitude components with

GM corrections in geometrized units ([5]) and performed the numerical simulations in two cases: a NS of $1.4M_{\odot}$ orbiting around a Super-MBH (e.g. Sagittarius A* 10^6M_{\odot}) and a BH of $10M_{\odot}$ orbiting around a Super-MBH. Computations are performed with orbital radii measured in units of the system reduced mass μ . Initial distances are sampled to show orbits from high eccentricity up to circularity ($e = \frac{r_{max}-r_{min}}{r_{max}+r_{min}}$). In Fig. 4-6 we show respectively the field velocities of the

Table 1. Data for GW for a NS of $1.4M_{\odot}$ orbiting around a Super-MBH

$\frac{r_0}{\mu}$	e	$f(mHz)$	h	h_+	h_{\times}
20	0.91	$7.7 \cdot 10^{-2}$	$2.0 \cdot 10^{-22}$	$5.1 \cdot 10^{-23}$	$5.1 \cdot 10^{-22}$
200	0.79	$1.1 \cdot 10^{-1}$	$1.2 \cdot 10^{-20}$	$2.2 \cdot 10^{-21}$	$3.1 \cdot 10^{-20}$
500	0.64	$1.4 \cdot 10^{-1}$	$6.9 \cdot 10^{-20}$	$8.7 \cdot 10^{-21}$	$1.7 \cdot 10^{-19}$
1000	0.44	$1.9 \cdot 10^{-1}$	$2.6 \cdot 10^{-19}$	$6.4 \cdot 10^{-20}$	$6.4 \cdot 10^{-19}$
1500	0.28	$2.3 \cdot 10^{-1}$	$4.8 \cdot 10^{-19}$	$3.6 \cdot 10^{-20}$	$1.2 \cdot 10^{-18}$
2000	0.14	$2.7 \cdot 10^{-1}$	$5.9 \cdot 10^{-19}$	$4.9 \cdot 10^{-20}$	$1.3 \cdot 10^{-18}$
2500	0.01	$3.1 \cdot 10^{-1}$	$5.9 \cdot 10^{-19}$	$1.7 \cdot 10^{-20}$	$9.2 \cdot 10^{-19}$

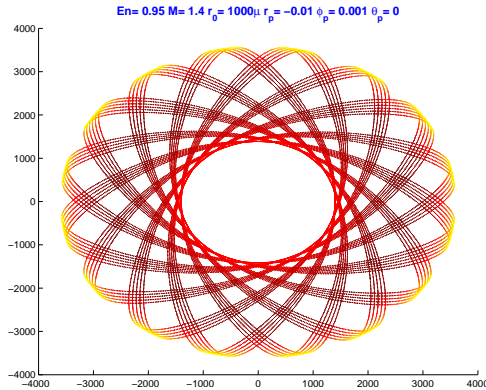


Figure 4. Plot of field velocities of the orbits.

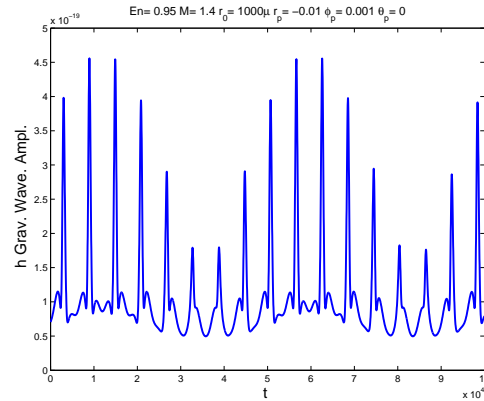


Figure 5. Total gravitational emission waveform h for a neutron star of $1.4M_{\odot}$

orbits along the axes of maximum covariances, the total gravitational emission waveform h and the gravitational waveform polarizations h_+ and h_{\times} for a NS of $1.4M_{\odot}$. The waveform were computed for the Earth-distance from Sagittarius A (central Galactic Black Hole). We obtained the numerical examples solving the ODE system for the following parameters and initial conditions: $\mu \approx 1.4M_{\odot}$, $E = 0.95, \phi_0 = 0, \theta_0 = \frac{\pi}{2}, \dot{\theta}_0 = 0, \dot{\phi}_0 = -\frac{1}{10}\dot{r}_0$ and $\dot{r}_0 = -\frac{1}{100}$ and $r_0 = (20 \div 1000)\mu$. See also Fig. 4-6 and Table 1. Finally we show in Fig. 7 the plot of the estimated h GW-strain-amplitudes for the considered binary sources at Galactic Center distance. The blue-line is the foreseen LISA sensitivity (one year integration + white dwarf background noise). The red diamonds ($1.4M_{\odot}$) and the green circles ($10M_{\odot}$) are the h values for the systems we have studied.

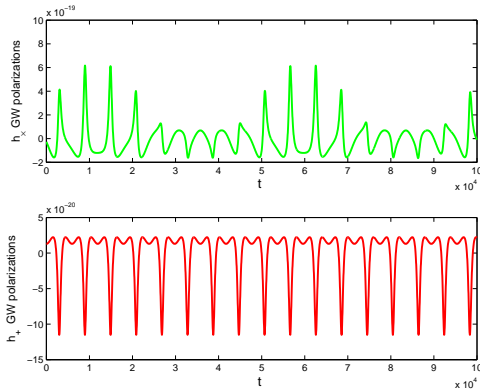


Figure 6. The gravitational wave-form polarizations h_+ and h_\times for a neutron star (NS) of $1.4M_\odot$

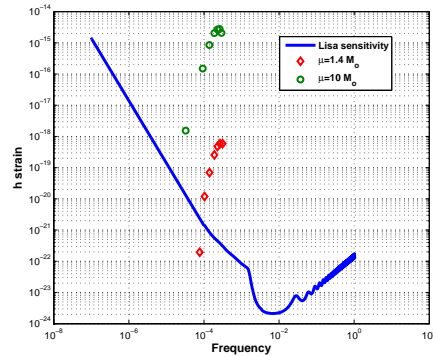


Figure 7. Plot of estimated mean values of gravitational emission in terms of strain h for two binary sources from the galactic center with reduced mass ratio $\mu \approx 1.4M_\odot$ (red diamonds) and $\mu \approx 10M_\odot$ (green circles). The blue line is the foreseen LISA sensitivity curve. The waveforms were computed for the Earth-distance to Sagittarius A (central Galactic Black Hole).

5. Concluding Remarks

The GM effect could give rise to interesting phenomena in tight binding systems such as binaries of massive objects (neutron stars or black holes). The effects appears particularly interesting if v/c is in the range $(10^{-1} \div 10^{-3})c$ as is, e.g. for objects captured by extremely massive black holes such as those at the Galactic Center. GM orbital corrections, after long integration time, induce precession and nutation effects capable of affecting the stability basin of the orbits. The global structure of such a basin is extremely sensitive to the initial radial velocities and angular velocities, the initial energy and masses which can determine possible transitions to chaotic behavior. In principle, GW emission could present signatures of GM corrections after suitable integration times in particular for the on going LISA space laser interferometric GW antenna.

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