The Physics of Planetary Ring Systems

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1 Introduction

Planetary ring systems are rich astrophysical environments that offer some of the most visually stunning solutions to Newton’s Laws in the solar system. Since the discovery of the Saturnian rings in 1610 by Galileo, planetary rings have been a frequent object of study by both physicists and astronomers. In the past thirty years enhanced earth based equipment and the scientific probes Pioneer-11, Voyager-1, Voyager 2 and Cassini have led to accelerated progress in the field, including the detection of ring systems about the three other gas giants in the Solar System: Neptune, Uranus and Jupiter. This, combined with the many analogies between planetary ring systems and other astrophysical objects such as accretion discs demonstrate a growing need to study such phenomena.

In this report we will be interested in the physics of these four ring systems, or more precisely, how physics is able to describe their formation, overall appearance and small scale structure. A thorough investigation must take into account the collective properties of the particles making up these rings and hence delve into the field of fluid dynamics; however, I will for the most part ignore these effects. Rather I will focus on the applications of classical mechanics to these ring systems, which has tremendous power in explaining most of the interesting features observed in the rings.

2 Comparative Anatomy of Planetary Ring Systems

Before discussing the physics, it is good to give an overview of the four main ring systems, which vary widely in their compositions and the some of the gross features of the rings themselves. Here I will briefly discuss each system, but because of its superlative nature, the most discussion here and in the rest of the document will be given to Saturn. Closest to Earth is the Jovian system, which is composed of rocky, dustlike particles organized in a thin main ring sandwiched between two thinner but more expansive rings (Murray et al. 1999, pp. 477). Because of the small particle sizes, the dynamics of these rings are governed mainly by electrodynamic and other nongravitational effects, which indicate a very short lifetime for the rings on the order of 1000 years (among other effects there is an inward force exerted on ring particles
Figure 1: The rings of Saturn (not to scale). The major rings are labeled by letters. Also marked is the Roche Limit (RL), the Cassini division (CD) and the moons Pan, Prometheus, Pandora and Mimas.

by the radiation from the sun, known as the Poynting-Robertson effect (Fridman et al. 1994, pp. 41).

Proceeding outward gives us the rings of Saturn, whose radial structure is shown in Figure 1 along with the orbital position of several important satellites. Each ring is labeled by a letter, nomenclature that reflects gaps in the rings and broad scale changes in ring composition. There is detailed structure on almost every spatial scale in the rings themselves, which further divides the rings into individual ringlets. The most prominent rings are the A and B rings, which are separated by the Cassini division, which is not actually a gap, just an area of reduced density (Murray et al., 1999 p. 478). Closer to the planet are the C and D rings which are optically thinner, and further away are the narrow F ring, which has an interesting braided structure, and the diffuse G and E rings.

The Uranian rings are a complex system composed of thirteen narrow, eccentric rings that are nearly perpendicular to Uranus’ orbital plane. The confinement of these rings poses interesting challenges for theorists, as rings are expected to both spread out and become less eccentric over time. Furthest from the sun is the Neptunian ring system. This system was first thought to be composed of several broken arcs instead of full rings, but these are now known to just be the denser part of a more extensive ring system. The strong azimuthal asymmetry has also posed an important question for theorists, one that has been given several explanations as we will discuss below.

3 Physics of Ring Systems

3.1 Preliminaries

A planetary ring may be imagined as a collection of small particles in the gravitational field of a massive planet. From Kepler’s first law, we know that the ring particles will travel in elliptical orbits with semi-major axis $a$ and eccentricity $e$: $r = \frac{a(1 - e^2)}{1 + e \cos (\theta - \bar{\omega})}$. 
where $\bar{\omega}$ is the angular position of the pericenter, or in astronomical terms, the longitude of pericenter. We will be considering orbits with low eccentricities ($e << 1$), and so this may be written to first order in $e$ as

$$r = a(1 - e \cos \theta - \bar{\omega})$$

To describe any such orbit in a fixed system rotating with the planet (Figure 2) we will need a total of six parameters: $a, e, I, \omega, \Omega, \tau$. We have accounted for $a$ and $e$, and $\tau$ is just initial time at which the object is at pericenter, the time of pericenter passage.

To draw this orbit with respect to a fixed frame, we may use the Euler angles, here with slightly different nomenclature. The orbit is inclined with respect to the fixed plane at an angle of inclination $I$ (in Goldstein's notation). This defines the orbital plane, which intersects the fixed frame at an angle $\Omega$, known as the longitude of the ascending node ($\phi$ for Goldstein). The line of intersection between the planes is known as the line of nodes. The angle between the line of nodes and the periapsis is called the angle of periapse, $\omega$. Therefore we have $\bar{\omega} = \Omega + \omega$ and we have fully defined the orbit of any body with respect to a fixed frame. We are also interested in the velocities of orbiting bodies. By Kepler’s third law, given a body in an elliptical orbit with semi-major axis $a$ and orbital period $T$ we have:

$$\frac{T}{2\pi} = \frac{a^3}{GM},$$

where $M$ is mass of the planet and $G$ is the gravitational constant. Using this we may write $n = \sqrt{\frac{GM}{a^3}}$, where $n$, the average angular frequency, is referred to as the mean motion.

3.2 The Guiding Center Approximation

The guiding center approximation is a regime in which one views an orbiting body in a reference frame rotating in a circle with angular frequency equal to the body’s
mean motion (This derivation follows (Murray et al., 1999 pp. 41-42)). More precisely, assume a situation as in Figure 3 where we have a satellite, S, in an elliptical orbit with semi-major axis $a$ about a planet at focus F. Take a reference frame centered about the imaginary particle $G$, known as the guiding center, and assume that $G$ is moving in a circular orbit with mean motion $n$. In the reference frame about the planet, the guiding center is at an angle $M = n(t - \tau)$ (the mean anomaly) and the particle is at an angle $f = \theta - \bar{\omega}$ (the true anomaly). From the figure, it is clear that at any given time, the position of the particle in a reference frame centered about $G$ is given by

$$x = r \cos(f - M) - a \quad y = r \sin(f - M)$$

Now the expansion of the term $f - M$ in terms of time is nontrivial, but can be done using Bessel functions as we have seen with the Kepler equation. To first order in $e$, $f - M \approx 2e \sin M$ (Murray 41). Further keeping terms to first order in $e$ we may take $r \approx a$, so that if we Taylor expand the sinusoidal functions and write $M = \kappa(t - \tau)$, we have

$$x \approx -ae \cos \kappa(t - \tau) \quad y = 2ae \sin \kappa(t - \tau)$$

From these equations we can see in the reference frame of the guiding center, the satellite moves in an ellipse with semi-major axis $2ae$ and angular velocity $\kappa$, which is known as the epicyclic frequency. Note that the epicyclic motion is directed opposite the velocity of the guiding center. While in this case, $\kappa = n$, this will not be the case when we handle the motion of particles about oblate spheroids where the mean motion and epicyclic frequency will differ, causing precession of the orbit. This method can be extended to orbits with some inclination, where the vertical excursions of a particle are given by a sinusoidal function with amplitude $I$ and frequency $\nu$.

### 3.3 Large Scale Ring Structure

Our first real application of physics to planetary rings will be to understand their large scale structure. (the derivation here follows (Esposito, 2006 pp. 38)). While this approximation is crude, and ignores any electromagnetic or thermodynamic effects, it gives good insight into the dynamics induced by the collisions of orbiting particles. Suppose we have a system of orbiting particles all of the same mass with orbits of varying
eccentricity and inclination. Collisions between particles in the system are not perfectly elastic, so energy must be lost. At the same time, we know from basic kinematics that the linear and angular momentum of the system must be conserved. Therefore the changes in velocity between the colliding particles must be equal in magnitude and oppositely directed. Since the energy depends quadratically on the velocity, the faster moving particle must move more slowly after the collision and vice versa. In order to conserve angular momentum about the planet, only the relative velocities of the particles are allowed to decrease. From a repeated application of these ideas, it is clear that orbiting particles will eventually be confined to a single plane and similar considerations show that eccentric motions will be damped out by collisions.

We can also use this model to account for the large radial spread of many ring systems. Suppose we have a ring confined to a plane with particles in nearly circular orbits. From above, we see that the rotational velocity of a particle in a circular orbit of radius $a$ is given by

$$v = nr = \sqrt{\frac{GM}{a}}$$

Thus there is a velocity gradient across the ring, known as the Kepler shear. When two particles with different radii collide, from above we see that their relative velocities will decrease. Hence the one rotating faster, the inner particle, will decrease in speed, and lose some angular momentum while the outer particle will increase in speed and gain angular momentum. Now $L = mvr = m\sqrt{GMr}$ in a steady state. Since the inner particle loses angular momentum, it must live at a smaller radius after the collision, and similarly the outer particle must live at a larger radius. Therefore the rings will gradually spread out in time. The exact speed at which they spread and whether they ever reach an equilibrium state are questions which go beyond the scope of this report.

### 3.4 The Roche Limit and the origin of Planetary rings

One important question that we have ignored until now is how planetary rings form. There are several competing theories for this, most of which invoke the idea of the Roche limit, defined as the radius at which the gravitational tidal forces from the planet exceed the self-gravitation of the satellite, causing it to break apart. This radius can be easily derived. To do so, let $R_p, M_p, R_s, M_s$ be the radii and masses of the planet and satellite, respectively and let $\rho_p, \rho_s$ be their respective densities. Take $a$ to be the distance between the planet and satellite. The Roche limit $a_r$ is the radius at which the tidal gravitational force on an infinitesimal element of mass $\Delta m$ exerted by the planet is equal to that exerted by the satellite:

$$G \frac{M_p \Delta m}{(a_r - R_s)^2} = G \frac{M_p \Delta m}{a_r^2} = G \frac{M_s \Delta m}{R_s^2}$$

Now $a_r \gg R_s$, so we may write the denominator of the left hand side as $[a_r (1 - \frac{R_s}{a_r})]^2$, and expand the term on the left as a geometric series to first order in the small term.
This gives $G \frac{M_p \Delta m R_s}{(a_s)^3}$ for the left hand side after identifying the zeroth order term with the traditional gravitational potential. Rewriting terms we have:

$$a_r^3 = 2 \frac{M_p R_s^3}{M_s} \text{ or } a_r = 2 R_p \left( \frac{\rho_p}{\rho_s} \right)^{1/3}$$

This theoretical limit, when coupled with the fact that most planetary rings are inside the Roche limit, points to one possible mechanism for ring formation: satellites and other large bodies that drift too close to the planet break apart, leaving behind small fragments that form planetary rings. While this model is attractive, it has some issues, as most small satellites are held together by forces much stronger than gravity (Fridman, 1994 p. 96). Accounting for these internal forces would yield a new, smaller Roche limit.

An alternative theory is that rings are formed from material left over the proto-planetary cloud that formed the system’s major satellites (Fridman, 1994 p. 98). While this too is an attractive theory, it cannot be complete, as most planetary ring systems either contain material that is much younger than the solar system or have ring features that can only exist on short timescales, such as the rings of Jupiter (Esposito, 2006 p. 104). To explain this, new theories are being developed which predict a continuous flow of material between moons and rings and point to a very dynamic picture of ring development (Esposito, 2006 p. 112).

### 3.5 Planetary Oblateness

It is well known that a rotating planet such as Saturn is not perfectly spherical, rather it is flattened at the poles, taking the form of an oblate spheroid with moments of inertia $I_3 > I_2 = I_1$. The gravitational potential of an oblate spheroid is therefore only axi-symmetric and may be written (Fridman et al., 1994 p. 72)

$$V = \frac{GM}{a} \left[ 1 - \sum_{n=2}^{\infty} J_n \frac{R}{a}^n P_n(\sin \phi) \right],$$

where here $R$ is the radius of the planet, the $J_n$ are constants known as the zonal harmonic coefficients, the $P_n$ are the Legendre polynomials, and $\phi$ is the angle between the planetary equator and the particle, which we will take equal to $\frac{\pi}{2}$ for rings close to the equatorial plane. Now the $J_n$ are rapidly decreasing functions for all planets (for Saturn $J_2 \approx 1.6 \cdot 10^{-2}$ while $J_4 \approx -1 \cdot 10^{-3}$), so we will only consider terms to order $J_2$, the quadrupole moment of the gravitational field. Now

$$P_2(\sin \phi) = \frac{1}{2} (3 \sin \phi - 1) = -\frac{1}{2},$$

and we may write our potential as a symmetric potential plus a perturbing term

$$U = U_0 + U', U_0 = \frac{GM}{6a}, U' = \frac{GMR^2}{2a^3} J_2$$
The full treatment of the effects of planetary oblateness involves a significant amount of perturbation theory, which would take up too much space to include here. A simple Lagrangian treatment treating just the change of pericenter is given in (Goldstein et al., 2002 pp. 223), while a Newtonian one is given in (Murray et al.,1999 pp. 268) and one involving Lagrange brackets is given in (Fridman et al., 2004 pp. 75). For our purposes, the effects of a perturbation are best seen using the guiding center approximation. The perturbing potential causes an increase in the orbiting body’s mean motion, a decrease in its epicyclic frequency and an increase in its vertical frequency (Murray et al., 1999 pp. 268):

\[ n^2 = n_0^2[1 + P] \]
\[ \kappa^2 = n_0^2[1 - P] \]
\[ \nu^2 = n_0^2[1 + 3P], \]

where \( P = \frac{3}{2} J_2 \frac{k^2}{a^2} \) and we have \( \nu > n > \kappa \), so that the epicyclic frequency and mean motion of the orbiting body are no longer equal. This implies a rotation of the orbit in space, and hence a precession of the pericenter with angular frequency

\[ \dot{\omega} = n - \kappa \approx n_0P \]

Because \( n > \kappa \), the perihelion advances. Similarly we have a change in the position of the line of nodes. This is similarly given by

\[ \dot{\Omega} = n - \nu \approx -n_0P \]

Since \( \nu > n \), the line of nodes moves in a direction opposite the perihelion, and hence regresses. While these effects are important in their own right, they have important implications when we consider gravitational resonances next.

### 3.6 Orbital Resonance & Density Waves

Up until now, we have developed a picture of planetary rings that accounts for their overall shape, but that does not explain the significant vertical, radial and azimuthal structure that is observed within the rings. Most of this structure can be understood by accounting for the gravitational effects of orbiting moons on ring particles. Like the forced oscillation of a pendulum, resonant effects will occur in the ring system when the small perturbing force of a moon is applied periodically: at the same point in the particle’s orbit over many orbits.

The simplest case of an orbital resonance is a co-rotation resonance, one in which the ratio between the orbital periods of the two particles is a simple fraction. One example is the Cassini division which occurs at the 2:1 co-rotation resonance of the Mimas satellite. This is an example of a destabilizing orbital resonance, one in which the gravitational perturbations cause particles to leave the resonance. Another is the Kirkwood gaps in the asteroid belt, which are explained by several resonances with the orbit of Jupiter. As this shows, orbital resonances are phenomena witnessed in many
places other than planetary rings. In fact, much of the theory we are about to discuss was developed to explain the structure of spiral galaxies.

To take a more rigorous approach to resonance, we must take into account all the characteristic frequencies of the orbiting moon and ring particles, which will be denoted \( n_s, \kappa_s, \nu_s \) and \( n_r, \kappa_r, \nu_r \), respectively. From this, we would expect there to be three types of resonance, one for each characteristic frequency. While a thorough development of the theory is beyond the scope of the report, we will briefly develop and give physical justification for the results, without going into all the mathematics (the derivation here follows Murray et al., 1999 pp. 482-485).

First consider the perturbing potential of the satellite \( U_s \), and its Fourier transform over all three frequencies. Define the pattern speed, \( \Omega_p \), to be the angular frequency of a reference frame that rotates with the perturbation:

\[
m \Omega_p = mn_s + k\kappa_s + p\nu_s,
\]

where \( m, k, p \) are non-negative integers. Using the results from above, this may also be written:

\[
m \Omega_p = (m + k + p)n_s - k\dot{\omega}_s - p\dot{\Omega}_s
\]

Co-rotation resonances occur when the ring particles are at rest in the frame of the perturbation, which when ignoring the changes in the pericenter and node occurs when the angular frequencies are indeed simple fractions:

\[
(m + k + p)n_s = mn_r
\]

Because we are in a frame rotating with speed \( mn_r \), we would expect there to be \( m \) regions in which resonance occur. Further because only the mean motion, \( n_r \) of the satellite is considered, the perturbation should only affect the semi-major axis of the ring particles, although the actual size of the variation is a bit beyond our scope. As a final note, the strongest resonances occur the smaller the difference between the periods is and the smaller \( m \) is, both of which are clear after a moment’s consideration.

A second type of resonance is known as Lindblad resonance. These occur when a ring particle feels the satellite’s perturbation periodically over its epicyclic orbit, and are thus defined by:

\[
m(n_r - \Omega_p) = \pm \kappa_r
\]

The positive negative signs correspond to the inner and outer Lindblad resonances (ILR, OLR), respectively. These occur depending on whether the perturbing satellite’s orbit is inside or outside the ring particle’s orbit. If we use the fact that \( \dot{\omega}_r = n_r - \kappa_r \) then as before we may write:

\[
(m + k + p)n_s - (m \mp 1)n_r - k\dot{\omega}_s \pm \dot{\omega}_r - p\dot{\Omega}_s = 0
\]

Ignoring the changes in pericenter and node, we see that resonances will be strongest when \( k = p = 0 \), and again the periods of the orbiting bodies must be close to simple
fractions for resonance to occur. An analogous relation holds for vertical frequencies, where we will have vertical resonances if

\[ m(n_r - \Omega_p) = \pm \nu_r \]

Just as co-rotation resonances modified the semi-major axis of the ring particles, we find that Lindblad and vertical resonances perturb the eccentricity and inclination of the particles, which will be discussed more below.

### 3.7 Density Waves

Now let's consider what Lindblad resonances look like in the moving frame. If we expand our orbital equation of an ellipse to first order in \( e \) as before, we have:

\[ r = a(1 - e \cos(\theta_r - \bar{\omega}_r)) \]

In the rotating frame, the angular position of a ring particle will be given by \( \theta'_r = \theta - \Omega_p t \). We may rewrite each side of our condition for Lindblad resonances as a total time derivative using \( \theta_r = n \) and \( \bar{\omega}_s = n - \kappa \). After integrating, and ignoring the constant, we obtain

\[ m\theta'_r = \theta_r - \bar{\omega}_r \quad \text{and so} \quad r = a[1 - e \cos m\theta'_r] \]

To first order in \( e \), the perturbed orbit of the particle in the moving frame is a stationary \( m \) lobed figure. For a destabilizing orbit of the particle in the moving frame is a stationary \( m \) lobed figure. For a destabilizing resonance, particles will be forced out of resonance with the satellite, and so we expect there to be \( m \) regions of the ring where an increase in density occurs. Now remember that there is differential rotation in the rings, so particles forced outside their previous orbit will move more slowly, those inside more quickly. If the Lindblad resonance is inside a ring, we then expect the density perturbation to take the form of a spiral coming outwards from the resonance. Further, if we take into account the self-gravity of the ring particles, this disturbance will propagate, giving rise to what is called a spiral density wave (Esposito, 2006 pp. 56). In the case of a vertical resonance, this is known as a spiral bending wave. If a Lindblad resonance occurs near the edge of a ring, we would expect there to be a sharp edge, which can be used to explain many of the sharp edges of the rings of Uranus.

### 4 Summary

The material reviewed here is by no means thorough or comprehensive, rather we have glossed over a significant body of mathematics in order to give simple, physical motivation for some of the properties of planetary rings. Conservation of momentum and energy mean that planetary rings should expand radially over time, while gravitational forces can offer a simple but limited view of how planetary rings form. Because rings orbit about an oblate spheroid, there are three characteristic frequencies to their motion, which in particular means that there are three different types of resonance that they may enter into with orbiting satellites.
These gravitational interactions between orbiting moons and planetary rings can explain a great deal of structure in planetary rings, far beyond what I have been able to cover here. In addition to causing resonances, moons embedded in rings cause gaps in the rings with wavy edges and moons can confine or shepherd rings, preventing their radial spread over time. There are many interesting examples of these in the rings of Saturn alone; Pan carves out the Encke gap in the rings of Saturn while the moons Prometheus and Pandora both confine the F ring and give particles within it chaotic orbits which have only recently been investigated. Gravitational interactions have also been used to explain the dense regions of Neptune’s rings, as well as the confinement of Uranus’s rings. All in all, investigating planetary ring system is a fruitful venture that exposes very elegant portions of classical mechanics across both large and small scales.

5 Bibliography