Landau damping, Stokes phenomenon and the weakly magnetized Maxwellian plasma

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Abstract

The dispersion relation for longitudinal oscillations in a weakly magnetized Maxwellian plasma is evaluated via a novel asymptotic method and reduces to the Bohm–Gross dispersion relation in the field-free limit. A proper treatment of the Stokes phenomenon reveals a factor of 2 missing in the standard result for Landau damping.

1. Introduction

Although the behaviour of a Maxwellian or classically thermal plasma in a strong magnetic field was presented by Bernstein in 1958 [1], the complementary behaviour in a weak magnetic field remains to this day an unsolved problem. This is despite the fact that the field-free behaviour, which serves as a limiting case for the latter system, had been evaluated in a seminal paper by Landau [2] more than a decade earlier.

In general, the physical behaviour of plasmas in weak magnetic fields including quantum mechanical systems such as the electron gas and charged Bose gas has remained elusive until a few years ago, when the properties of the weakly magnetized charged Bose gas were evaluated for the first time at $T = 0 \text{ K}$ [3]. This was accomplished by employing a novel asymptotic expansion for the incomplete gamma function presented in Ref. [4], which appears in all the system’s response properties [5]. Furthermore, Appendix D of Ref. [3] indicated that the approach used to obtain the novel expansion could be adapted to determine the physical properties of a weakly magnetized Maxwellian plasma. Therefore, the aim of this Letter is to study longitudinal oscillations, also known as plasma oscillations, in a weakly magnetized Maxwellian plasma. In so doing, we shall revisit the field-free limit and re-examine Landau damping of the oscillations. With the aid of a relatively new
numerical technique known as Mellin–Barnes regularization, which yields exact values for divergent series more quickly and accurately than existing methods, we shall see that previous derivations of the damping have omitted a factor of 2, a discrepancy that has arisen because the Stokes phenomenon has not been handled properly.

2. Dispersion relation for the magnetized Maxwellian plasma

This section begins with a summary of the derivation of the dispersion relation for longitudinal oscillations in a magnetized Maxwellian plasma before presenting the final equation. In order to study the dispersion relation in the weakly magnetized limit, Bernstein’s formalism [1] was used in preference to the more modern and elegant treatments of Ichimaru [6] and Swanson [7] because the latter produce intractable infinite series over the orders of Bessel functions for the dispersion relation. To solve the resulting equation given in these references, drastic assumptions have to be employed, which invariably means that only the zeroth order terms are retained in the weak magnetic field limit. The situation is not as bad for the strong magnetic field limit. The situation is not as bad for the strong magnetic field limit. The situation is not as bad for the strong magnetic field limit. The situation is not as bad for the strong magnetic field limit. The situation is not as bad for the strong magnetic field limit.

From the Fourier–Laplace transforms of Maxwell equations, Bernstein obtains a vector equation given as Eq. (20) in Ref. [1]. This equation possesses the tensor or dyadic \( Q \), which is related to the conductivity tensor in Ref. [6] and is given by

\[
Q = s \sum_j \pm \frac{4\pi Z_j^2 e^2}{m_j} \int d^3 \mathbf{v} \int_{-\infty}^{\infty} d\Omega_j \frac{G}{\Omega_j} \frac{\partial f_0(\mathbf{v}')}{\partial \mathbf{v}'} \mathbf{v}.
\]  

The sum in the above result is over all species of charged particles present in the plasma. By using this result one obtains the dispersion relation for longitudinal oscillations from

\[
k^2 \omega^2 + \mathbf{k} \cdot \mathbf{Q} \cdot \mathbf{k} = 0,
\]

where nowadays \( s \) is replaced by \( i\omega \) and our primary aim is to solve for \( \omega \) as a function of the wavenumber \( k \). If the Maxwellian distribution function of

\[
f_0(u) = \frac{n_j}{\sqrt{\pi} v_{T_j}} e^{-u^2/v_{T_j}^2},
\]

where \( n_j \) and \( v_{T_j} (= \sqrt{2 k T_j/m_j}) \) are respectively the number density and thermal velocity for species \( j \) introduced into Eq. (1), then \( Q \) for a multi-component plasma in equilibrium can be calculated after a lengthy calculation. Introducing the ensuing result into Eq. (2) yields the dispersion relation for the plasma oscillations, which is

\[
k^2 + \sum_j k_{Dj}^2 = \sum_j \frac{i\omega k_{Dj}^2}{\Omega_j} \int_0^\infty dy \exp(-iy/\Omega_j) \times \exp(-\mu_j y^2/2 - \lambda_j (1 - \cos y)),
\]

where \( k_{Dj}^2 (= 4\pi n_j e_j^2/k T_j) \) is the square of the Debye wavenumber and \( \Omega_j (= |e_j|B/m_j c) \) is the cyclotron frequency for species \( j \). Also in Eq. (4), \( \mu_j = k_\perp^2 v_{T_j}^2/2\Omega_j^2 \) and \( \lambda_j = k_\parallel^2 v_{T_j}^2/2\Omega_j^2 \) and \( \mathbf{k} \) is the propagation vector whose components parallel and perpendicular to the direction of the magnetic field are \( k_\parallel \) and \( k_\perp \), respectively. Eq. (4) is basically a generalization of Eq. (37) in Ref. [1], which is, in turn, basically the result obtained by Gordeyev in Ref. [8]. This form for the dispersion relation is seldom used by plasma physicists and so, the exponential term with the cosine function is generally replaced by an infinite series over the order of modified Bessel functions. However, we shall write this term as

\[
\exp(\lambda (\cos y - 1 + y^2/2)) = \sum_{m=0}^\infty (-1)^m y^m d_m(\lambda),
\]

where as explained in Appendix A, the \( d_m(\lambda) \) are special polynomials that can be determined by adapting the graphical technique presented in Refs. [3,4,9].

Table 1 presents the first nine \( d_m(\lambda) \). It can be seen that for \( m \geq 2 \), the \( d_m(\lambda) \) are of order \( [m/2] \) where \([x]\) represents the greatest integer less than or equal to \( x \). Furthermore, since \( d_m(\lambda) = \sum_{j=1}^{[m/2]} d_m^j \lambda^j \) for \( m \geq 2 \).
one finds that the two lowest and the highest order coefficients are given by
\[
d^m_{1} = 1/(2m)!,
\]
\[
d^m_{2} = (2^{m-2} - 1 - m(2m - 1))/(2m)!,
\]
and
\[
d^m_{m/2} = \frac{(1 + (-1)^{m})/2}{(m/2)!(4!)^{m/2}}\bigg[ (1 - (-1)^{m})/2 \cdot 6! + \frac{1}{((m - 3)/2)!(4!)^{(m-3)/2}} \bigg].
\]
If Eq. (5) is introduced into Eq. (4), then the resulting integrals can be evaluated by using Ref. [11]. The dispersion relation for plasma oscillations in a \( j \)-component magnetized Maxwellian plasma becomes
\[
k^2 + \sum_{j} k^2_{Dj} = \sum_{j} \frac{i \sqrt{\pi} \omega k_{Dj}}{k v T_j} \left[ f(\omega/k v T_j) + \sum_{m=2}^{\infty} \Omega_{jm}^{2m} d_m(\lambda_j) \frac{\partial \omega^{2m}}{\partial \omega^m} f(\omega/k v T_j) \right],
\]
where \( f(z) = \exp(-z^2) \text{erfc}(z) \). The first term in square brackets on the rhs of Eq. (7) represents the field-free result, while the magnetic field only appears in the product of \( \Omega_{jm}^{2m} \) and \( d_m(\lambda_j) \) in the second term. For \( \lambda_j \gg 1 \) or \( k^2_{Dj} \gg \Omega_{jm}^{2m} \), \( \Omega_{jm}^{2m} d_m(\lambda_j) \) is \( O(\lambda_j) \) when \( m = 2k \) and it is \( O(\Omega_{jm}^{2k}) \) when \( m = 2k + 1 \). For \( \lambda_j \ll 1 \), which can occur when \( k \approx k_z \) or the propagation vector of the oscillations is almost parallel to the direction of the magnetic field, \( \Omega_{jm}^{2m} d_m(\lambda_j) \) is \( O(\Omega_{jm}^{2m-2}) \). Since the above analysis has been carried out for a multi-component plasma, this means that collective modes can be determined for a two component electron–ion plasma, which will become the subject of a future publication. However, from here on in this Letter, we shall only be concerned with the one-component electron plasma. Therefore, the subscripts \( j \) and \( e \) will be dropped, where unnecessary.

For an electron plasma \( \omega \) is expected to be at least the size of the plasma frequency \( \omega_p = \sqrt{4\pi n_e e^2/m} \) and much greater than \( k v_T \). This means that in order to solve Eq. (7), a large \( |z| \) series expansion is required for the function, \( f(z) \), which is often written in terms of the plasma dispersion function \( Z(z) \) [12] or the \( W \)-function of Ref. [6].

### 3. Standard asymptotics

According to Dingle [13], the large \(|z| \) asymptotic expansion for the error function can be written as
\[
\text{erf}(z) = 1 - \text{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} \sum_{l=0}^{\infty} \frac{\Gamma(l + 1/2)}{(-z^2)^l}.
\]
In equivalence (8), \( \sigma \) equals 1, 0, or -1 corresponding to whether \(|\arg z| < \pi/2, \arg z = \pi/2 \) or \( \pi/2 < |\arg z < 3\pi/2 \). The reader should note the appearance of the ‘\( ^\pm \)’-operator on the right-hand side, which is due to the fact that the asymptotic series is divergent, unlike the convergent series given in Eq. (5), which possesses an ‘\( ^\mp \)’-operator on the right-hand side. Since the series in Eq. (8) is divergent, for it to yield meaningful results, it must be regularized as discussed in Refs. [9,10]. Furthermore, the result is non-uniform because \( \sigma \) varies across the positive imaginary axis, which is known as a Stokes line of discontinuity [13].

For small values of \(|z| \), i.e., \(|z| < 1 \), the asymptotic series does not possess an optimal point of truncation [14], and thus, truncating the series expansion in Eq. (8) will not yield accurate approximations for the error function. On the other hand, if \(|z| \) is large, i.e., \(|z| \gg 1 \), then the optimal point of truncation represents the first value of \( k \), where two successive terms in the series are almost equal to one another. In addition, for \(|z| > 5 \), the optimal point of truncation increases with increasing \(|z| \). When an asymptotic series is truncated

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Table 1

<table>
<thead>
<tr>
<th>( m )</th>
<th>( d_m(x) ) in Eq. (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>( x/4! )</td>
</tr>
<tr>
<td>3</td>
<td>( x/6! )</td>
</tr>
<tr>
<td>4</td>
<td>( x/8! + x^2/2 \cdot 4!/2 )</td>
</tr>
<tr>
<td>5</td>
<td>( x/10! + x^2/4! \cdot 6! )</td>
</tr>
<tr>
<td>6</td>
<td>( x/12! + 29x^2/2 \cdot 5! + 81x^3/6 \cdot 4!/2 )</td>
</tr>
<tr>
<td>7</td>
<td>( x/14! + 2x^2/5! \cdot 9! + x^3/2 \cdot 4!/2 \cdot 6! )</td>
</tr>
<tr>
<td>8</td>
<td>( x/16! + 1251x^2/8! \cdot 11! + 43x^3/4! \cdot 6! \cdot 81 + x^4/(4!/4) )</td>
</tr>
</tbody>
</table>
at its optimal point, the remainder attains its minimum value, once it has been regularized. However, for $|z| > 5$, the truncated series will yield an accurate approximation to the error function even if only a few terms are considered in the truncated series. That is,

$$\text{erf}(z) = 1 - \text{erfc}(z) \approx \sigma - \frac{e^{-z^2}}{\pi z} \sum_{l=0}^{N_T} \frac{\Gamma(d + 1/2)}{(-z^2)^l},$$  \quad (9)

where $N_T < N_0$ and $N_0$ is the optimal point of truncation. If this approximation with $\sigma = 0$ is introduced into $f(z)$, then one obtains the dispersion relation for longitudinal oscillations in a field-free Maxwellian plasma as given in Refs. [6,7], which is

$$1 + \frac{\tilde{\omega}^2}{\tilde{\pi}^2} Z(\tilde{\omega}) \approx 1 - \tilde{\omega}^2 \sqrt{\frac{\omega k}{D}} \exp \left( - \frac{\sigma^2}{\sqrt{8} k v_T} \right).$$  \quad (10)

In Eq. (10) the tilde notation, e.g., $\tilde{z}$, denotes $z/k v_T$, which will be used throughout this work. The solution to approximation (10) is obtained by putting $\omega = \omega_r + i \gamma_i$, thereby yielding

$$\omega_r^2 \approx \omega_p^2 + \frac{3}{2} k^2 v_T^2 + \cdots,$$  \quad (11)

and

$$\gamma_i \approx -\omega_p \sqrt{\frac{\pi}{8} k^3} \exp \left( - \frac{\sigma^2}{\sqrt{8} k v_T} \right).$$  \quad (12)

The result for $\omega_r$ is known as the Bohm–Gross dispersion relation [15]. Furthermore, even though the ‘$\approx$’-operator appears, we are still considering extremely accurate evaluations of the physical quantities, provided $|\tilde{\omega}| > 5$.

Although Vlasov [16] derived approximation (10) first, he neglected the imaginary term because he assumed that $\gamma_i$ was positive [7]. On the other hand, it is often claimed that by treating the problem as an initial value problem, Landau [2] was able to overcome this inconsistency. Hence, plasma oscillations in an unmagnetized Maxwellian plasma are said to be Landau damped, which is an effect that results from the resonant interaction of a wave with free-streaming particles. As stated by O’Neill and Coroniti [17], high temperature plasmas are collisionless to a first approximation and as a consequence, this gives rise to new phenomena such as Landau damping, which would be masked by collisions in a normal fluid. For a brief summary of early studies of Landau damping and a discussion of its presence in hydrodynamics, astrophysics and other systems, the reader is referred to Ref. [18].

4. Mellin–Barnes regularization

The problem with approximation (12) is that it has been obtained by putting $\sigma$ equal to zero in equivalence (8). According to Dingle [13], however, the appropriate form when $\pi/2 < \arg i \omega < 3\pi/2$, i.e., when $\gamma_i$ is negative, is equivalence (8) with $\sigma = -1$. Thus, the problem is whether the correct value of $\sigma$ in the asymptotic expansion can be determined when moving across the positive imaginary axis or Stokes line. Such a calculation will require summing the divergent series for values of $z$ near the positive imaginary axis. Normally, one would choose large values of $|z|$ and truncate the series, but because the exponential factor outside the series dominates for large imaginary values, the contribution due to $\sigma$ in equivalence (8) will be concealed for small deviations from the imaginary axis. Hence, we shall have to consider small values of $|z|$ near the positive imaginary axis and not truncate the series in equivalence (8).

The most famous regularization technique for evaluating asymptotic series is Borel summation, which, when applied to the series in Eq. (8), would yield a terminating integral as discussed in Chapter 21 of Ref. [13]. Unfortunately, when numerically evaluating the terminating integral via a numerical integration routine such as NIntegrate in Mathematica [19], one finds that the accuracy can be affected by the $z$-dependent pole in the denominator, especially for small $|z|$. On other occasions, when requiring machine precision accuracy, the process of evaluation can be very slow.

The entire series, however, can be summed by using the remarkable technique of Mellin–Barnes (MB) regularization, which was first introduced in Ref. [20]. In this approach, a divergent series can be made equivalent to a particular MB integral, which can also be evaluated numerically with the aid of the NIntegrate routine in Mathematica. Since its first appearance, where it was used to gain exact values from the asymptotic expansions for an exponential series of the
form, $S_p(a) = \sum_{k=0}^{\infty} \exp(-ak^p)$, the technique has been employed in a study of the Bender–Wu asymptotic formula for the ground state energy of the anharmonic oscillator [21] and in providing exact results for Bessel and Hankel functions from their asymptotic expansions [22]. It is also much simpler to implement, more general and more accurate than the awkward technique of hyperasymptotics [23,24] that has been adopted by the asymptotics community over the past decade or so [25]. As a consequence, it is seen as a vital tool in the development of a theory of divergent series [10].

For a complex power series defined by $S(N, z) = \sum_{k=N}^{\infty} f(k)(-z)^k$, if there exists a real number $c$ such that the poles of $\Gamma(N-s)$ lie to the right of the line $N - 1 < \text{Re} s = c < N$ and the poles of $f(s)\Gamma(1 + s - N)\Gamma(N-s)$ are single-valued to the right of the line, $S(N, z)$ is equivalent to the following MB integral:

$$S(N, z) = \frac{e^{-z}}{\pi z} \int_{c-i\infty}^{c+i\infty} ds \frac{z^s f(s)}{e^{\pi s} - e^{-\pi s}}.$$  (13)

Therefore, MB regularization of the divergent series in the asymptotic expansion for the error function yields

$$S(0, z^2) = -\frac{e^{-z^2}}{\pi z} \sum_{k=0}^{\infty} \Gamma(k + 1/2) \left( -\frac{1}{z^2} \right)^k$$

$$\equiv \frac{e^{-z^2}}{\pi z} \int_{c-i\infty}^{c+i\infty} ds \frac{z^{-2s} \Gamma(s + 1/2)}{(e^{\pi s} - e^{-\pi s})}.$$  (14)

As will be seen shortly, equivalence (14) requires modification when $z$ lies on a Stokes line of discontinuity, which according to Chapter 1 of Ref. [13], is whenever $z$ is purely imaginary, i.e., $\arg z = \pm \pi/2$.

To demonstrate the power of equivalence (14), we shall consider two typical cases, the first where $|z|$ falls in the intermediate range, viz. $|z| \approx 2$, and the second, where $|z|$ is small, i.e., $|z| \ll 1$. In the first case, the series will be evaluated for $z$ equal to $0.05 + 2i$, $2i$ and $-0.05 + 2i$, while in the second case $z$ will equal $0.05 + 0.05i$, $0.05i$ and $-0.05 + 0.05i$. Table 2 presents the values obtained for both cases using Mathematica [19] on a Pentium III computer.

The second column in Table 2 presents the actual values for the error function, while the third displays the results by applying the numerical integration routine called NIntegrate to the rhs of equivalence (14) with $c$ equal to $-1/4$. It can be seen that when $z$ is equal to $0.05 + 2i$ and $0.05 + 0.05i$, $\sigma$ must equal unity for the MB values to yield the corresponding values of the error function. For $z$ equal to $-0.05 + 2i$ and $-0.05 + 0.05i$, $\sigma$ must equal $-1$ for the MB values to yield the corresponding error function values. However, when $z$ is purely imaginary, $\sigma$ must also equal $-1$ for the two sets of results to agree, instead of zero as postulated by Dingle. When adding the appropriate values for $\sigma$ to their respective MB results, it was found that the values for the error function and the MB values agreed to 13 significant figures for all values of $z$. Furthermore, many other cases than those presented in Table 2 were considered, but all yielded results consistent with the behaviour in the table.

The reason why $\sigma$ must equal $-1$ instead of zero for the MB results to agree with the actual values on the imaginary axis or Stokes line is due to the multivaluedness of $(-1)^s$ in the MB integral of equivalence (14). Evidently, Mathematica selects only one of the two values for $(-1)^s$, whereas according to

<table>
<thead>
<tr>
<th>$z$</th>
<th>erf(z)</th>
<th>MB value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.05 + 2i$</td>
<td>3.057341631949816 + 18.25817377997483i</td>
<td>2.057341631949817 + 18.25817377997484i</td>
</tr>
<tr>
<td>$0 + 2i$</td>
<td>0 + 18.56480241457555i</td>
<td>1.000000000000001 + 18.56480241457556i</td>
</tr>
<tr>
<td>$-0.05 + 2i$</td>
<td>-3.057341631949816 + 18.25817377997483i</td>
<td>-2.057341631949817 + 18.25817377997484i</td>
</tr>
<tr>
<td>$0.05 + 0.05i$</td>
<td>0.0565128487368875 + 0.05632478587819847i</td>
<td>-0.943487151263113 + 0.05632478587819847i</td>
</tr>
<tr>
<td>$0 + 0.05i$</td>
<td>0 + 0.05646600943625297i</td>
<td>1.000000000000002 + 0.0564660094362528i</td>
</tr>
<tr>
<td>$-0.05 + 0.05i$</td>
<td>-0.0565128487368875 + 0.05632478587819847i</td>
<td>0.943487151263113 + 0.05632478587819847i</td>
</tr>
<tr>
<td>$2i$</td>
<td>given above</td>
<td>0.0 + 18.56480241457559i</td>
</tr>
<tr>
<td>$0.05i$</td>
<td>given above</td>
<td>0.0 + 0.05646600943625293i</td>
</tr>
</tbody>
</table>
p. 13 of Ref. [13], it should select the average of the two values. That is, whenever a Stokes line is encountered, the factor of $(-1)^k$ occurring in the numerator of the MB integral should be replaced by $(\exp(i\pi s) + \exp(-i\pi s))/2 = \cos(\pi s)$. Therefore, for $z = iy$, equivalence (14) should be modified to

$$S(0, z^2) = \frac{e^{\pi y}}{i\pi y} \int_{-\infty}^{c+\infty} ds \, y^{-2} \frac{\Gamma(s + 1/2)}{\Gamma(s)} \frac{\cos(\pi s)}{e^{\pi s} - e^{-i\pi s}}. \quad (15)$$

The results for $z$ equal to $2i$ and $0.05i$ when applying NIntegrate to equivalence (15) appear at the bottom of Table 2, where it can be seen that the exact values for the error function have now been evaluated. Hence, by modifying the MB integral in equivalence (14) to take into account the Stokes line, one finds that $\sigma$ equals zero. Therefore, the three different asymptotic forms for the error function near the positive imaginary axis as displayed by equivalence (8) are valid.

From the first case in Table 2 we see that the asymptotic series in Eq. (8) dominates the $\sigma$ term when $z$ is near the positive imaginary axis and $|z| > 1$. However, the subdominant $\sigma$ term is sizable near this Stokes line when $|z| < 1$. Although the subdominant term in the asymptotic expansion for the error function is not composed of a divergent series as is often the case, the extremely accurate results in Table 2 show clearly that a discontinuity exists across a Stokes line. This behaviour is in agreement with the numerous and more complicated asymptotic expansions in Ref. [9], where the subdominant terms were composed of divergent series, but is in conflict with the claim that there should be a smoothing of the Stokes phenomenon along the positive imaginary axis as described in Ref. [26].

If approximation (9) is introduced into the field-free part of equivalence (7), then the dispersion relation for an electron Maxwellian plasma becomes

$$1 - \frac{k_D^2}{k^2} \sum_{l=1}^{N_T} \frac{\Gamma(l + 1/2)}{\Gamma(l/2)\omega^{2l}} = -i \sqrt{\bar{\omega}} \frac{k_D^2}{k^2} (1 - \sigma) e^{\omega^2} \approx 0. \quad (16)$$

The larger $|\omega|$ is, the larger $N_T$ can become. In addition, the above approximation becomes more accurate. From the above numerical analysis, when $\arg(i\omega) < \pi/2$, $\sigma$ is equal to unity and the imaginary contribution to the dispersion relation vanishes. Hence, $\gamma_1$ is equal to zero, when it should not be. Thus, no consistent solution exists for $\arg(i\omega) < \pi/2$. For $\arg(i\omega) = \pi/2$, $\sigma = 0$. Then one obtains Landau’s result for $\gamma_1$, viz. approximation (12), which contradicts the condition that $\omega$ must be real. Hence, only when $\sigma$ equals $-1$ or $\pi/2 < \arg(i\omega) < 3\pi/2$, can there be a solution, which means damping of the modes. Putting $\omega = \omega_r + i\gamma$ and $\sigma = -1$, one finds that $\omega_r$ is the same as in approximation (11), but that $\gamma_1$ is given by

$$\gamma_1 \approx \omega_P \sqrt{\frac{\pi k_D^3}{2 k^3}} \exp(-\omega_r^2), \quad (17)$$

or a factor of 2 greater than the result given by approximation (12).

It might be argued that Malmberg and Wharton, who report the results of an experiment measuring the damping of plasma oscillations in Ref. [27], have verified the Landau damping formula. However, a close inspection of Fig. 3 in this paper reveals that only two of the thirteen measurements carried out by them actually fall on the curve obtained from the theory of Landau, even though the results are presented on a log-linear scale. One of these measurements tips the curve at the edge of its error bar, while the other represents the point used to calibrate the measurements, which is not correct experimental procedure anyway. All other measurements can easily account for a factor of 2 and cannot compete in accuracy with the results presented in Table 2. Thus, one can only conclude that this experiment has verified the existence of Landau damping, which is not in dispute here, but it definitely cannot claim to have validated the formula as derived by Landau.

The factor of 2 discrepancy is certainly a strange anomaly, bearing in mind that the Stokes phenomenon was discovered more than 80 years [28] before Landau damping. That Landau was unaware of the phenomenon until at least 1957, more than a decade after his seminal paper on the damping of plasma oscillations, has been documented by Pokrovskii’s account of his meeting with Landau [29] on the subject of above-barrier reflection [30], which was the earliest use of the method of matched asymptotic expansions in the complex plane. For an understanding of the role of the Stokes phenomenon in this method, the reader is referred to Chapter 4 of Ref. [31], while p. 567 of this
reference discusses Landau’s lack of understanding of the phenomenon.

5. Landau’s derivation

Now that we have established that there is a discrepancy of a factor of 2 in the formula for Landau damping of the unmagnetized Maxwellian plasma, we need to indicate where the error has arisen in Landau’s seminal paper. To do so, we must examine how Landau obtained the asymptotic form for the induced potential \( \phi(t) \) for large values of the time \( t \) in Ref. [2]. First, he considers the inverse Laplace transform of

\[
\phi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dp \phi_pe^{pt},
\]

where the integration is performed along a line contour in the right-hand half of the complex plane and \( \phi_p \) is given by

\[
\phi_p = \frac{4\pi e}{k^2} \int_{-\infty}^{\infty} du \frac{g(u)/\left(p+iku\right)}{1-4\pi e^2 \int_{-\infty}^{\infty} du \frac{df_0(u)}{du}/\left(p+iku\right)}.
\]

In Eq. (19), \( g(u) = \int dv_x dv_\perp g(v) \), where \( u = v_x \) and \( g(v) \) is the Fourier component of the initial velocity distribution function, viz. \( f(r,v,0) \). The appearance of \( g(u) \) in the above result is the reason why Landau’s derivation of the damping is referred to as an initial value approach, which, as mentioned previously, is often cited as the reason why Vlasov was unable to observe damping. As we shall see, this has nothing to do with the Vlasov’s inability to obtain damping since \( g(u) \) does not appear in the dispersion relation. In addition, \( f_0(u) \) is the equilibrium or Maxwellian distribution function given by Eq. (3). To make contact with the previous material such as Eq. (4), we note that \( p = -i\omega - \gamma \).

From Laplace transform theory, the dominant contribution to Eq. (18) for large values of \( t \) comes from the poles of \( \phi_p \), which are obtained by setting the denominator equal to zero. This gives

\[
I = \frac{4\pi i e^2}{km} \int \frac{du}{c} \frac{df_0}{(p+iku)} = 1,
\]

where the integral is now interpreted as a Cauchy integral with the contour being the real axis. So far, all is in order, but the following steps by Landau are dubious. Without a reference to a specific theorem, he claims that when \( Re p < 0 \), one can choose a point \( A \) on the real axis situated not far from the pole \( u = -p/ik \) so that a semi-circle can be drawn through the pole. Then he claims that the modified contour can be used instead of \( C \). Finally, he claims that the integral along the semi-circle yields a semi-residue contribution, which is the term responsible for Landau damping. On p. 9 of his book [32], Sazhin refers to this claim as an assumption, which is used in later chapters to analyze more complicated equations for electromagnetic wave propagation in a magnetized plasma. Since we have seen from the analysis of the asymptotic form for the error function in the previous section that the damping of plasma oscillations for the unmagnetized Maxwellian plasma violates this assumption by a factor of 2, the more complicated examples of damping studied by Sazhin must also be incorrect.

Let us examine the Cauchy integral in Eq. (20) in more detail. This integral is proper provided \( arg(ip/k) \neq 0 \). Initially, Landau’s derivation was valid for \( Re p > 0 \), which means that since \( p = -i\omega - \gamma \), the singularity at \( u = -ip/k \) lies in the second quadrant of the complex plane. Of course, there is no damping when the integral is evaluated, which is consistent with Vlasov’s finding. The concern is what happens when \( p \) is analytically continued to \( Re p < 0 \). For \( Re p = 0 \), the integral is undefined as the singularity lies on the contour, while for \( Re p < 0 \), the singularity now lies in the third quadrant. This is the type of situation described on p. 413 of Ref. [33]. As discussed in Ref. [9], in order to obtain the correct values for the contour integral in Eq. (20) when singularities move across the contour, we need to include their residue contribution. In fact, from the more sophisticated examples presented in this reference, a rule has been developed that when the pole moves in an anti-clockwise direction for an asymptotic series in inverse powers of \( z \), i.e., large \( |z| \), the semi-residue contribution from the Cauchy integral must be subtracted from the asymptotic expansion when \( Re p = 0 \). In addition, if MB regularization is performed on the asymptotic series, then factors of \( \exp(\pm i\pi s) \) must be replaced by \( \cos(s\pi) \), while if the series is regularized by Borel summation, then the Cauchy principal value of the resulting integrals must be taken. For \( Re p < 0 \), although the Cauchy integral is defined, we must sub-
tract the full residue contribution in order to obtain the correct values for the original integral. If the movement had been in a clockwise direction, then instead of subtracting the contributions, we would need to add them. Therefore, these lead to the appearance of jump discontinuities in the asymptotic forms, first found by Stokes in 1857. The residue at \( u = ip/k \) for the Cauchy integral in Eq. (20) is

\[
\text{Res}(I) \bigg|_{u=ip/k} = \frac{2np}{\sqrt{\pi}k^2v_T^2} e^{p^2/k^2v_T^2}.
\]

(21)

From the preceding discussion, for \( \text{Re}\ p < 0 \), the dispersion relation given by Eq. (20) becomes

\[
\frac{4\pi i e^2}{km} \left( \frac{1}{p} \int_{-\infty}^{\infty} \frac{du}{1+iku/p} \frac{d\omega}{i} - 2\pi i \text{Res}(I) \right) = 1.
\]

(22)

Because the contour integral is defined, its limits have been re-introduced. To obtain an asymptotic series for the above integral, we interpret the denominator as the regularized value for the geometric series. That is,

\[
\sum_{j=0}^{\infty} \left( \frac{iku}{p} \right)^j \equiv \frac{1}{1+iku/p}.
\]

(23)

The derivation of this result is presented in Ref. [10]. Introducing the geometric series into Eq. (22) gives

\[
\frac{\omega^2}{p^2} \sum_{j=0}^{\infty} (-1)^{j+1} \frac{\Gamma(j+3/2)}{\Gamma(3/2)} \left( \frac{k^2v_T^2}{p^2} \right)^j + 4i\sqrt{\pi} \frac{\omega^2 p}{k^2v_T^2} e^{p^2/k^2v_T^2} \equiv 1.
\]

(24)

The ‘\( \equiv \)’ operator has been introduced indicating that equivalence (24) needs to be regularized. As explained in Ref. [10], regularization is necessary in order to correct the impropiety of the asymptotic method. Since \( \omega^2/v_T^2 = k^2/2 \), the above equivalence yields approximation (16) when the divergent series is truncated at \( N_T \) and \( p \) is set equal to \( i\omega \). Assuming that \( \gamma \) is very small. If \( \text{Re}\ p = 0 \), then we would obtain the same result except the leading term would be a factor of two smaller. This would lead to the Landau damping result except the solution would violate the condition on \( p \) being purely real.

6. The weakly magnetized Maxwellian plasma

The magnetic field terms in Eq. (7) involve multiple derivatives of \( f(z) \). With the aid of No. 7.1.19 in Ref. [34] and No. 8.950 in Ref. [35], each multiple derivative yields a finite sum containing products of Hermite polynomials \( H_m(z) \). However, much cancellation occurs such that the final result is identical to differentiating directly the asymptotic expansion for \( f(z) \), which is obtained by using equivalence (8) instead. Therefore, the dispersion relation given by Eq. (7) becomes

\[
\frac{k^2}{k^2_B} + \frac{1}{\omega^2} = \sum_{m=0}^{\infty} \frac{(\tau e)^{2m}}{\omega^2} \frac{d\nu_m}{\omega} \sum_{l=0}^{\infty} \frac{\Gamma(2l+2m+1)}{\Gamma(l+1)(2\omega)^{2l}}
\]

\[
+i \sqrt{\pi}(1-\sigma)\omega e^{-\omega^2}
\]

\[
\times \sum_{m=0}^{\infty} d_m(\lambda_t)\Omega_t^{2m} H_{2m}(\omega).
\]

(25)

To solve equivalence (25), we need to truncate the divergent series on the rhs. The inner series in the first quantity on the rhs is of a similar form to the asymptotic series for the error function, which can be seen by introducing the duplication formula for the gamma function for the \( m = 0 \) term. This series can be truncated if \( \omega \gg 1 \). However, even if the inner series is truncated, we are still left with a divergent series, which can be written as

\[
S = \sum_{m=0}^{\infty} \frac{\Omega_t^{2m}}{\Gamma(2m+1) + \frac{\Gamma(2m+3)}{(2\omega)^2} + \cdots}.
\]

(26)

By introducing the first and third results of Eq. (6) into the above, we see that \( S \) can only be truncated if \( |\Omega_t/\omega| \ll 1 \) and \( |\Omega_t/\sqrt{\pi}\omega|\sqrt{24} \ll 1 \).

Since \( |\omega| \gg 1 \), \( H_{2m}(\omega) \approx 2^m\omega^{2m} \). Then the series in the second quantity of the rhs of equivalence (26) can be written as

\[
\sum_{m=0}^{\infty} d_m(\lambda_t)\Omega_t^{2m} H_{2m}(\omega)
\]

\[
\approx \sum_{m=0}^{\infty} d_m(\lambda_t)(2\omega^{2m})^{2m}
\]

\[
= \exp(\lambda\omega(\cosh(2\omega^{2m}) - 1 - 2\omega^{2m})),
\]

(27)
where in obtaining the final form, we have replaced \( y \) by \( 2i\tilde{\omega}\tilde{\Omega}_r \) in Eq. (5). Therefore, we see that the second quantity on the rhs of equivalence (25) is convergent as in the free-field situation. That is, truncation is required only when evaluating the dispersion relation for the modes and not for the damping.

If we apply the preceding material conditions and results to equivalence (25), then we find that

\[
\frac{k^2}{k_B^2} + 1 \approx \sum_{m=0}^{N_1} \left( \frac{\tilde{\omega}_r}{\tilde{\omega}} \right)^{2m} \frac{\Gamma(2l + m + 1)}{\Gamma(l + 1)(2\tilde{\omega})^{2l}} \left[ i\sqrt{2}(1 - \sigma)\tilde{\omega}_r e^{-2\tilde{\omega}} \times \exp(\tilde{\lambda}_e F(\tilde{\omega}_r, \tilde{\Omega}_e)) \right],
\]

(28)

where \( F(x, y) = \cosh(2xy) - 1 - 2k^2y^2 \) and \( N_1 \) and \( N_2 \) represent the optimal points of truncation for the divergent series and may not necessarily be equal to one another. \( N_1 \) becomes larger, the larger \( |\tilde{\omega}| \) is, while \( N_2 \) becomes larger, the larger \( |\tilde{\omega}/\tilde{\Omega}_r| \) and \( |\tilde{\omega}\sqrt{24}/\tilde{\Omega}_r\sqrt{\lambda_e}| \) are. Note also that \( 2\tilde{\Omega}_r/\tilde{\omega} \ll 1 \) ensures that the final term on the rhs of equivalence (28), which represents the contribution due to damping of the modes, will be small.

We have seen that by putting \( \sigma \) equal to unity, we obtain the Vlasov result. Although the Vlasov result is unable to yield the damping of the modes, it nevertheless yields the dispersion relation for the modes. Therefore, putting \( \sigma \) equal to unity in approximation (28) for the time being and invoking the above conditions so that \( N_1 \) and \( N_2 \) are much greater than unity, we obtain the following solution to second order:

\[
\omega_{p, r}^2 \approx \omega_p^2 + \frac{3}{2} k^2 v_T^2 + \frac{k^2}{k_B^2} \tilde{\omega}_r^2 + \frac{3}{2} k^4 v_T^4 + \frac{k^2 v_T^2}{k_B^2} \tilde{\omega}_r^2 + \frac{3}{2} k^4 \tilde{\omega}_r^4 + \frac{9}{2} \frac{\Omega_e^2}{\omega_p^2} k^2 \tilde{\omega}_r^2 + \cdots.
\]

(29)

The first two terms on the rhs of approximation (29) represent the Bohm–Gross dispersion relation, while the next term when combined with the first term represents the dispersion relation for a cold magnetoplasma. Since it contains the ratio of \( k_\perp/k \), it is, as expected, anisotropic. From the above dispersion relation we that the weakly magnetized plasma behaves like a classical Maxwellian plasma when the modes propagate almost parallel to the direction of the magnetic field. In addition, the cyclotron motion of the particles only affects the modes when the direction of propagation has a substantial perpendicular component. The fourth and fifth terms on the rhs of approximation (29) represent the second order terms for a Maxwellian plasma and a cold magnetoplasma respectively, while the final term is a by-product of the first order terms for both these plasmas. Now if we let \( \sigma = -1 \) and \( \tilde{\omega} = \tilde{\omega}_r + i\tilde{\gamma}_r \), where \( |\tilde{\gamma}_r/\tilde{\omega}_r| \ll 1 \), then we find that

\[
\frac{\gamma_1}{\omega_r} \approx \frac{2\sqrt{\pi} \omega_p^3 e^{-\omega_p^2}}{1 + 3/\omega_p^2 + 2k_B^2 \tilde{\Omega}_r^2 / k_B^2 \tilde{\omega}_r^2 + 45/4\omega_p^4}.
\]

(30)

In the limit \( B \to 0 \), this result reduces to approximation (17).

An interesting result can also be obtained by considering the case, where \( \lambda_e \ll 1 \), in addition to \( kv_T \ll \omega \). In this case, either \( kv_T \ll \Omega_e \) or the direction of propagation is almost parallel to the magnetic field. Then equivalence (25) yields

\[
1 = \frac{\omega_p^2}{\omega_r^2} \left( 1 + 3/2\omega_r^2 + O(\omega_r^4) \right)
+ \lambda_e \left[ \frac{k_B^2}{k^2} g_1(\tilde{\Omega}_e/\tilde{\omega}) \right]
+ \frac{\omega_p^2}{\omega_r^2} \left[ \frac{3}{2} \omega_r^2 g_3(\tilde{\Omega}_e/\tilde{\omega}) \right] + O(\omega_r^6)
+ i\sqrt{\pi}(1 - \sigma) \frac{k_B^2 \tilde{\omega}_r}{k_B^2 \tilde{\omega}_r} e^{-2\tilde{\omega}}
\times \left[ 1 + \lambda_e (e^{-2\tilde{\omega}} \cosh(2\tilde{\omega} \tilde{\Omega}_e) - 1 + \tilde{\Omega}_e^2 - 2\omega_p^2 \tilde{\Omega}_r^2 \right] + O(\lambda_e^2),
\]

(31)

where \( g_1(z) = z^3/(1 - z^2), \ g_2(z) = (1 + 3z^2)/(1 - z^2)^3 - 1 - 6z^2, \) and \( g_3(z) = (1 + 10z^2 + 5z^4)/(1 - z^2)^5 - 1 - 15z^2 \). Eq. (31) has been obtained by introducing the first result in Eq. (6). This produces variants of the geometric series, which are easily regularized according to equivalence (23). The situation is more complicated if we proceed to the next order in \( \lambda_e \), in which case the second result in Eq. (6) would need to be introduced. This produces divergent series, which can be regularized by Borel summation, but yield unwieldy forms. In Eq. (31) the damping term has been obtained by using the generating function for Hermite polynomials, viz. No. 8.957(1) in Ref. [35]. Now if we put \( \sigma = -1 \) and perturb around the plasma frequency,
then we find that
\[
\omega_r^2 \approx \omega_p^2 \frac{3}{2} k^2 v_T^2 + \frac{k^2}{2} \frac{\Omega_e^2}{\omega_p^2} \frac{1}{2} k^2 \frac{\Omega_e}{\omega_p} \left[ \frac{k^2}{k^2} g_1 \left( \frac{\Omega_e}{\omega_p} \right) + g_2 \left( \frac{\Omega_e}{\omega_p} \right) \right] + \cdots,
\]
and
\[
\frac{\gamma}{\omega_r} \approx A e^{-\omega_p^2} \left( 1 + \lambda_e \left( e^{-\omega_p^2} \cosh(2\omega_p\Omega_e) \right) - 1 + \Omega_e^2 - 2\omega_p^2\frac{\Omega_e^2}{\omega_p^2} \right),
\]
where \( A = \sqrt{\pi k^2 \omega_p / k^2} \).

For modes that do not propagate almost parallel to the magnetic field, the first order contribution due to the magnetic field will be greater than the first order thermal term. In fact, for \( \omega_r^2 \gg \Omega_e^2 \gg k^2 v_T^2 \), the magnetic field contribution is dominated by the \( g_1(\Omega_e/\omega_p) \) term, which means that the dispersion relation reduces to
\[
\omega_r^2 \approx \omega_p^2 \frac{3}{2} k^2 v_T^2 + \frac{k^2}{k^2} \frac{\Omega_e^2}{\omega_p^2} \frac{1}{2} \cdots,
\]
Therefore, we have obtained the dispersion relation given by approximation (29) except now the final term due to a cold magnetoplasma dominates the second or thermal plasma term for modes not propagating almost parallel to the direction of the magnetic field. Otherwise, the plasma behaves much like a field-free Maxwellian plasma. Likewise, the damping of the modes given by approximation (33) reduces to that for a field-free Maxwellian plasma, i.e., approximation (17), for modes which propagate almost parallel to the direction of the magnetic field.

Appendix A. Calculation of the polynomials \( d_m(x) \)

To calculate the polynomials \( d_m(x) \) given in Eq. (5), we adapt the method presented in Refs. [3,4,9], which entails constructing a tree diagram for each polynomial. For a value of \( m \) one draws branches to all pairs of numbers that can be summed to \( 2m \), where the first number equals zero or is any even number between 3 and \( m \). E.g., for \( d_7(x) \) there are three branch lines leading to \( (0, 14), (4, 10) \) and \( (6, 8) \). If the first number of a couple is zero as in \( (0, 14) \), then there are no further branches from that couple and the path terminates. For the other pairs, the same prescription is applied to the second value in the couple as to the initial value of \( 2m \). Hence, for \( (4, 10) \) branch lines are drawn to \( (0, 10) \) and \( (4, 6) \), while for \( (6, 8) \) branch lines are drawn to \( (0, 8) \) and \( (4, 4) \). The process continues recursively until each path in the tree diagram terminates with a couple possessing a zero as its first member.

This, however, is not all that is needed to produce the final tree diagram for each \( d_m(x) \). The first member of each couple along a path in the tree diagram and the second member of the terminating couple form a partition for \( 2m \). E.g., the path comprising of \( (4, 10), (4, 6) \) and \( (0, 6) \) is a representation of the partition \( [4, 4], 6 \), while the path with \( (6, 8) \) and \( (0, 8) \) represents the partition \( [6, 8] \). If in the process of creating the tree diagram for \( d_m(x) \), a partition is duplicated such as \( (6, 8), (4, 4) \) and \( (0, 4) \), then this path should be removed from the diagram. That is, each partition must appear only once in each diagram. Thus, the final tree diagram for \( d_7(x) \) possesses only four distinct paths.

Each path or partition in a tree diagram yields a contribution to \( d_m(x) \) based on the elements in the partition. Let \( l_i \) and \( n_i \) be respectively the elements and the frequency each element appears in a particular partition. If \( N \) represents the total number of elements in a partition, then \( \sum n_i = N \). The contribution that such a partition makes to \( d_m(x) \) is \( x^N / (l_1)^{n_1} \cdots (l_i)^{n_i} \cdots n_i! \cdots n_i! \). For example, the contribution of the partition \( [4, 4, 6] \) to \( d_7(x) \) is simply \( x^7 / 6! \cdot 4!^3 \cdot 2 \), while the partition \( [4, 10] \) contributes \( x^3 / 4! \cdot 10! \). Hence, we see that the number of elements in a partition or the number of branches along a path yield the power of \( x \) in \( d_m(x) \), while all partitions for which \( \sum n_i = N \) contribute to the final value for the coefficient of \( x^N \) in \( d_m(x) \). Thus, there are four separate contributions to \( d_7(x) \) from the partitions in the tree diagram with the partitions \( [4, 10] \) and \( [6, 8] \) combining to give the second order term in \( x \).

References

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