Standard Cosmology

- The fate of the universe
- Red shift
- Hubble law
- Age of the universe
- Problems of standard cosmology
Long-term behaviour of $a(t)$ (I)

The full solutions to the Friedmann equations can be quite complicated. However, it is possible to study a few generic case, in particular starting from the present time ($\Rightarrow$ radiation plays no role) and considering $t \to \infty$. In case $\Lambda = 0$,

$$\rho = \rho_m = \rho_0 \frac{a_0^3}{a^3}$$

$$\dot{a}^2 = \frac{8\pi G_N a_0^3 \rho_0}{3 a} - k$$

$k = 0$  \hspace{1cm} $a(t) \approx t^{2/3}$

$k = -1$  \hspace{1cm} $\lim_{t \to \infty} a(t) \approx t$

$k = +1$  \hspace{1cm} $a_{\text{max}} = \frac{8\pi G_N a_0^3 \rho_0}{3}$

Einstein-de Sitter universe

since at the end the matter term becomes negligible

imposing that the right member be 0
Long-term behaviour of a(t) (II)

In case \( \Lambda \neq 0 \),

\[
\ddot{a} = \frac{8\pi G_N a_0^3 \rho_0}{3a} - k + \frac{\Lambda a^2}{3}
\]

\( \Lambda \) contribution is negligible for small \( a(t) \) and dominates for large \( a(t) \). This is puzzling: why is it that we live at a so special epoch, when \( \rho_\Lambda \) is of the order of \( \rho_m \)?

\( \Lambda < 0 \)

\[
\frac{8\pi G_N a_0^3 \rho_0}{3a_{\text{max}}} - k + \frac{\Lambda a_{\text{max}}^2}{3} = 0
\]

\( \Lambda > 0 \) \( k = 0, -1 \)

\[
a(t) \equiv a(t_0)e^{\sqrt{\Lambda/3}(t-t_0)}
\]

\( \Lambda > 0 \) \( k = 1 \)

there exists \( \Lambda_c \) giving a static solution,

\( \Lambda > \Lambda_c \Rightarrow \) expansion forever,

\( \Lambda < \Lambda_c \Rightarrow \) forbidden range for \( a(t) \)
Solutions of the Friedmann equation

\[ \Omega_M = 0.2/\Omega_\Lambda = 0.8 \]
\[ \Omega_M = 0.2/\Omega_\Lambda = 0 \]
\[ \Omega_M = 1.0/\Omega_\Lambda = 0 \]
\[ \Omega_M = 2.0/\Omega_\Lambda = 0 \]

\[ \Lambda = 0, \ k = -1 \]
\[ \Lambda = 0, \ k = 0 \]
\[ \Lambda = 0, \ k = +1 \]

\[ t_H = H_0^{-1} \]
Redshift (I)

Let us consider the geodesic motion of a particle that is not necessarily massless,

\[ \frac{d u^\mu}{d \lambda} + \Gamma^\mu_{\nu \alpha} u^\nu \frac{d x^\alpha}{d \lambda} = 0 \]

where \( u^\mu = dx^\mu / ds \) and \( \lambda \) is some affine parameter. If \( \lambda \) is the proper length, \( s \), the geodesic equation becomes

\[ \frac{1}{u^0} \frac{d |\vec{u}|}{ds} + \frac{\dot{a}}{a} |\vec{u}| = 0 \]
Redshift (II)

Since $u^0 = dt/ds$, we get

\[ \frac{\ddot{u}}{|\dot{u}|} = -\frac{\dot{a}}{a} \]

that is the three-momentum of a freely-propagating particle in an expanding universe red shifts. This occurs for photons too,

\[ |\vec{u}| \propto a^{-1} \]

\[ |\vec{p}| = m |\vec{u}| \propto a^{-1} \]

The redshift $z$ is often used in place of the scale factor.
Redshift (III)

Consider a fixed comoving volume of the universe, i.e. a volume specified by fixed values of the coordinates, from which one may obtain the physical volume at a given time $t$ by multiplying by $a(t)^3$. Given a fixed number of dust particles (of mass $m$) within this comoving volume, the energy density will then scale just as the physical volume, i.e. as $a(t)^{-3}$, in agreement with the equation of state with $w = 0$.

In the case of radiation, because of the redshift, the energy density of a fixed number of photons in a fixed comoving volume drops with the physical volume (as for dust) and by an extra factor of the scale factor, as the expansion of the universe stretches the wavelengths of light. Thus, the energy density of radiation will scale as $a(t)^{-4}$, once again in agreement with the equation of state with $w = 1/3$. 

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Linear Hubble’s law

In a uniformly expanding universe, in first approximation, Hubble’s law is valid,

\[ v = H d \]

Let us consider a bidimensional analogy. The distance between two points on a 2-sphere at \( t_1 \) and \( t_2 \) is \( d_1 = a(t_1)s \) \( d_2 = a(t_2)s \), where \( s \) is the arclength distance between the points. Then

\[ v = \lim_{t_2 \to t_1} \frac{d_2 - d_1}{t_2 - t_1} = \lim_{t_2 \to t_1} \frac{a(t_2) - a(t_1)}{t_2 - t_1} s = \dot{a} s = H d \]
Deviations from linear Hubble’s law

Actually, acceleration equation implies that, if matter drives the expansion, the rate is presently decelerating.

However, observations indicate that vacuum energy is dominating and the expansion is accelerating. In any case, we may expand the scale factor in a Taylor series around the present time, \( t_0 \),

\[
a(t) = a(t_0) \left( 1 + H_0 (t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + ... \right)
\]

which defines the deceleration parameter, \( q_0 \).

As \( a_0 = a(t_0) \), we have

\[
q_0 = - \frac{\ddot{a}_0}{a_0 H_0^2}
\]
Deceleration parameter

From the acceleration equation, the expression of $\rho_c$, and the equation of state for each different contribution (matter, radiation, and vacuum) to the universe energy density, we get

$$q_0 = \frac{1}{2} \frac{8 \pi G_N}{3 H_0^2} \sum_i (\rho_{i0} + 3 p_{i0}) = \frac{1}{2} \sum_i \Omega_{i0} (1 + 3 \omega_i)$$

Since a cosmological constant has $\omega=-1$, $q_0$ can be negative if the energy density of the universe is dominated by $\rho_\Lambda$. In general, acceleration of the expansion is possible for $\omega_i=-1/3$. 
Red-shift expansion

Using the definition of red-shift, the expansion of \( a(t) \) gives

\[
1 + z = \frac{a_0}{a(t)} = \left( 1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + \ldots \right)^{-1}
\]

\[
1 + z = 1 - H_0(t - t_0) + \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + H_0^2(t - t_0)^2 + \ldots
\]

\[
1 + z = 1 - H_0(t - t_0) + \left( 1 + \frac{q_0}{2} \right) H_0^2 (t - t_0)^2 + \ldots
\]

\[
z = -H_0(t - t_0) + \left( 1 + \frac{q_0}{2} \right) H_0^2 (t - t_0)^2 + \ldots
\]
Particle horizon

A light ray that moves radially \((\theta, \phi=\text{const})\) toward us, from \(r=r_e\) at time \(t_e\) to \(r=0\) at time \(t\), along a geodesic \((ds^2=0 \Rightarrow dt=a\, dr/(1-kr^2)^{1/2})\), will satisfy

\[
\int_0^{r_e} \frac{dr}{\sqrt{1-kr^2}} = \int_{t_e}^{t} \frac{dt'}{a(t')}
\]

and the physical distance will be

\[
d(t) = a(t) \int_0^{r_e} \frac{dr}{\sqrt{1-kr^2}} = a(t) \int_{t_e}^{t} \frac{dt'}{a(t')}
\]

Particle horizon is the farthest physical distance we can observe in the past. It is given by the limit \(t_e \rightarrow 0\), unless the integral does not converge (this does not happen in conventional cosmological theories). In a RD era, \(d_p=2t\), while in a MD era \(d_p\approx3t\).
Event horizon

While particle horizon is the maximum distance wherefrom signals can reach the observer, event horizon is the maximum distance that signals we send can reach. It is determined by

\[ \int_{r_0}^{\infty} \frac{dr}{\sqrt{1-kr^2}} = \int_{t}^{\infty} \frac{dt'}{a(t')} \]

In a matter dominated era, the integral diverges, so that there is no event horizon. But, in presence of a cosmological constant, \( a(t) \) will eventually grow exponentially, giving a finite event horizon. This means that, as time passes, all sources of light outside our Local Group will move beyond \( d_e \) and become unobservable.
Horizons

Let us remember the definition of particle and event horizon:

\[ d_p(t) = a(t) \int_0^t \frac{dt'}{a(t')} \]
\[ d_e(t) = a(t) \int_t^\infty \frac{dt'}{a(t')} \]

If \( a(t) \sim t^\alpha \), then

\[ \alpha < 1 \quad \Rightarrow \quad d_p(t) = \frac{t}{1-\alpha} \quad d_e(t) = \infty \]

\[ \alpha > 1 \quad \Rightarrow \quad d_p(t) = \infty \quad d_e(t) = \frac{t}{\alpha-1} \]

If \( a(t) \sim e^{Ht} \), then

\[ d_p(t) = \frac{e^{Ht} - 1}{H} \quad d_e(t) = \frac{1}{H} \]

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Distance expansion

Expanding left and right member of previous equation gives

\[
\frac{1}{\sqrt{1-kr^2}} = 1 + \frac{kr^2}{2} + \ldots
\]

\[
\int_0^{r_e} \frac{dr}{\sqrt{1-kr^2}} = r_e + \frac{k}{6}r^3_e + \ldots
\]

\[
\frac{1}{a(t)} = \frac{1}{a_0} \frac{a_0}{a(t)} = \frac{1}{a_0} \left( 1 - H_0(t-t_0) + \left(1 + \frac{q_0}{2}\right)H_0^2(t-t_0)^2 + \ldots \right)
\]

\[
\int_{t_e}^{t_o} \frac{dt}{a(t)} = \frac{1}{a_0} \left( t_o - t_e \right) + \frac{H_0}{2} \left( t_o - t_e \right)^2 + \ldots
\]

and finally

\[
r_e = \frac{1}{a_0} \left( t_o - t_e \right) + \frac{H_0}{2} \left( t_o - t_e \right)^2 + \ldots
\]
Luminosity distance (I)

To obtain information on the cosmological parameters we use measurements like the intensity of a source of known intrinsic strength, a *standard candle*. Given a source of absolute luminosity $L$, its apparent luminosity at a distance $d$ is $I = L / (4\pi d^2)$. At large distance this expression needs modification. Suppose that $N_\gamma$ photons was emitted isotropically at $t_e$. If there were no expansion, a telescope of area $A$, at distance $a(t_e)r$, would see a fraction of photons,

$$\frac{N_{\text{det}}}{N_\gamma} = \frac{A}{4\pi(a(t_e)r)^2}$$

Due to the expansion, at detection time, $t_o$,

$$\frac{N_{\text{det}}}{N_\gamma} = \frac{A}{4\pi(a(t_o)r)^2}$$
Luminosity distance (II)

Two additional effects have to be taken into account. First, all photons will have energy red-shifted by the factor $1+z$. Second, the rate of arrival of individual photons is lower than the rate at which they are emitted, since the time interval will be increased by the same factor, $1+z$. This means that the flux measured at the telescope will be

$$\mathcal{F} = \frac{L}{4\pi (a(t_o)r)^2 (1+z)^2} = \frac{L}{4\pi d_L^2}$$

which defines the luminosity distance, $d_L$,

$$d_L = a(t_o)r(1+z)$$
Hubble’s law

After some arithmetics,

\[ z = H_0 (t_o - t_e) + \left(1 + \frac{q_0}{2}\right) H_0^2 (t_o - t_e)^2 + \ldots \]

\[ t_o - t_e = \frac{1}{H_0} \left(z - \left(1 + \frac{q_0}{2}\right) z^2 + \ldots \right) \]

\[ r_e = \frac{1}{a_0} \left(t_o - t_e + \frac{H_0}{2} (t_o - t_e)^2 + \ldots \right) \]

\[ r_e = \frac{1}{a_0 H_0} \left(z - \frac{1}{2} (1 + q_0) z^2 + \ldots \right) \]

\[ d_L = \frac{1}{H_0} \left(z + \frac{1}{2} (1-q_0) z^2 + \ldots \right) \]

\( z \sim v \)

deviation from the linear Hubble law
Standard candles

Establishing the distances to other galaxies has been one of the main efforts of observational cosmology. For minimizing the uncertainties introduced by peculiar velocities, one needs to reach the Hubble flow, that is the distance range where the velocities of the galaxies are dominated by the cosmological expansion. To do that, one of the most common methods exploits the homogeneity of the peak brightness of Type Ia supernovae (SNIa), which has about a 20% intrinsic dispersion. Connecting the distance of the galaxy hosting the supernova with the recession velocity (given by the red-shift), one builds a Hubble diagram.

SNIa: a small star (white dwarf) which, accreting matter from a companion, overcomes the Chandrasekar limit
Hubble diagram

Hubble diagram for high red-shift SNIa from the *Supernova Cosmology Project* and low red-shift SNIa from the *CfA* and *Calan/Tololo Supernova Survey*. The data favour a universe with a positive cosmological constant. The analysis of SNIa gives:

\[
h = 0.65 \pm 0.02 (\pm 0.05)\]

\[
q_0 = -0.65 \pm 0.15
\]

\[
q_0 = \frac{1}{2} (\Omega_M - 2\Omega_\Lambda)
\]
Degeneracy in the SNIa analysis

The SNIa data are insufficient to break the degeneracy of the density terms. The results can be approximated by the linear combination

\[ 0.8 \Omega_M - 0.6 \Omega_\Lambda = -0.2 \pm 0.1 \]

Such a degeneracy can be break only with the inclusion of other kind of observation.
Hubble parameter evolution

Remembering the rate of change of $\rho_M$ and $\rho_\Lambda$ with $a(t)$, the definition of red-shift and curvature parameter,

$$\rho = \rho_M + \rho_\Lambda = \rho_M^0 \left( \frac{a_0}{a} \right)^3 + \rho_\Lambda^0 = \rho_c^0 \left( \Omega_M^0 (1 + z)^3 + \Omega_\Lambda^0 \right)$$

$$- \frac{k}{a^2} = - \frac{k}{a_0^2} \left( \frac{a_0}{a} \right)^2 = H_0^2 \Omega_k^0 (1 + z)^2$$

$$H^2 = H_0^2 \left( \Omega_M^0 (1 + z)^3 + \Omega_\Lambda^0 + \Omega_k^0 (1 + z)^2 \right)$$
Age of universe

From the definition of the Hubble parameter and red-shift,

\[
H = \frac{d}{dt} \left( \ln \frac{a}{a_0} \right) = -\frac{d}{dt} \left( \ln(1 + z) \right) = -\frac{1}{1 + z} \frac{dz}{dt}
\]

\[
\frac{dt}{dz} = -\frac{1}{1 + z} \frac{1}{H} = -\frac{1}{H_0} \frac{1}{\left( \Omega_M^0 (1 + z)^3 + \Omega_\Lambda^0 + \Omega_k^0 (1 + z)^2 \right)^{1/2}}
\]

and the present age of the universe is

\[
t_0 = \frac{1}{H_0} \lim_{z \to \infty} \int_0^z \frac{dz'}{(1 + z')(\Omega_M^0 (1 + z')^3 + \Omega_\Lambda^0 + \Omega_k^0 (1 + z')^2)^{1/2}}
\]
Age of a flat universe

\[ \Omega_M = 1 \quad \Omega_\Lambda = 0 \quad \Omega_k = 0 \]

For \( H_0 = 65 \text{ km s}^{-1} \text{ Mpc}^{-1} \), this gives an age of about 10 billion years.

The age of the universe for a given \( \Omega_M \) is always larger in the flat case (with a cosmological constant). Also, the age is larger for smaller values of \( \Omega_M \).
The flatness problem

From the Friedmann equation,

\[ H^2 = \frac{8\pi G_N}{3} \rho - \frac{k}{a^2} \]

\[ \frac{3H^2}{8\pi G_N \rho} = 1 - \frac{k}{8\pi G_N \rho a^2} \]

\[ \Omega^{-1} - 1 = -\frac{3k}{8\pi G_N \rho a^2} \approx a^{\alpha-2} \]

This shows that at early times (small \(a\)) \(\Omega\) must have been very close to unity. For example, at the time of nucleosynthesis: \(\Omega(t=1\text{s})=1\pm10^{-16}\) and at the Planck time \(\Omega(t=10^{-43}\text{s})=1\pm10^{-60}\).

\[ \Omega^{-1} - 1 \Bigg|_{T_{Pl}} \approx \left( \frac{a}{a_0} \right)^2 = \left( \frac{T_0}{T_{Pl}} \right)^2 \approx 10^{-60} \]

How is it possible a similar fine-tuning?
The horizon problem

Let us consider two photons which arrive to us today from two different directions in the sky. It is possible to consider directions corresponding to regions causally decoupled at the time of decoupling.

How is it possible that the two photons have the same $T$?
The $\Lambda$ problem

Going back to the complete Friedmann equation, we see that the comparison between the matter term and the cosmological constant term is given by their ratio, $\kappa$,

\[ H^2 = \frac{8\pi G_N}{3} \rho - \frac{k}{a^2} + \frac{\Lambda}{3} \]

\[ \kappa = \frac{\frac{\Lambda}{8\pi G_N \rho}} \]

Considering the value that $\kappa$ has today, one obtains that at the Planck time it should have been $10^{-122}$!!!!

How is it possible to have a so small dimensionless number?