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Exploring the Boundary of Half-Positionality

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Abstract Half positionality is the property of a language of infinite words to admit positional winning strategies, when interpreted as the goal of a two-player game on a graph. Such problem applies to the automatic synthesis of controllers, where positional strategies represent efficient controllers. As our main result, we present a novel sufficient condition for half positionality, more general than what was previously known. Moreover, we compare our proposed condition with several others, proposed in the recent literature, outlining an intricate network of relationships, where only few combinations are sufficient for half positionality.

1 Introduction

Games are widely used in computer science as models for describing multi-agent systems, or the interaction between a system and its environment [8, 10, 12, 13]. Usually, the system is a component that is under the control of its designer and the environment represents all the components that the designer has no direct control of. In this context, a game allows the designer to check whether the system can force some desired behavior (or avoid an undesired one), independently of the choices of the other components. Further, game algorithms may automatically synthesize a design that obtains the desired behavior.

We consider games played by two players on a finite graph, called an *arena*. The arena models the interaction between the entities involved: a node represents a state of the interaction, and an edge represents progress in the interaction. We consider turn-based games, i.e., games where each node is associated with only one player, who is responsible for choosing the next node. A sequence of edges in the graph represents a run (or *play*) of the system. Player 0 wants to force the system to follow an infinite run with a desired property, expressed as a language of infinite words called a *goal*.

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The objective of player 1 is the opposite. We distinguish *uninitialized* arenas, where the game can start from any node, from *initialized* ones, which include a fixed initial node.

In this context, a *strategy* for a player is a predetermined decision that the player makes on all possible finite paths ending with a node associated to that player. A strategy is *winning* for a player if it allows him to force a desired path no matter what strategy his opponent uses. A key property of strategies is the amount of memory that they require in order to choose their next move. The simplest strategies do not need to remember the past history of the game, i.e., their choices only depend on the current state in the game. Such strategies are called *positional*.

We are interested in determining the existence of a winning strategy for one of the players, and possibly compute an effective representation of such a strategy. To this aim, suitable techniques have been developed when the desired behavior of a player is specified in specific forms, such as temporal logic [8] or the parity condition [3, 10, 11]. In those applications where the aim is to effectively generate a winning strategy, only positional strategies may be suitable for a concrete implementation, due to space constraints. In fact, notice that even a positional strategy, which is a function from states to moves, needs an amount of storage that is proportional to the size of the state-space of the system. Symbolic representations can mitigate such issues [1]. For this reason, it is useful to know when a given goal guarantees that if player 0 (respectively, player 1) has a winning strategy then he has a positional one. This property is called *half-positionality* (in the following, HP) for player 0 (resp., player 1). If a goal is HP for both players, the goal is called *full positional* (FP). Notice that HP is more important than FP in the synthesis applications we are referring to. In these applications, player 0 represents the controller to be synthesized and player 1 the environment. Hence, we are only interested in obtaining simple winning strategies for one of the two players, namely for player 0.

Full positionality has been studied and characterized: in [3, 10, 11], it was proved that the parity winning conditions are full positional and in [4] Gimbert and Zielonka present a complete characterization for full positional determined goals on finite unitialized games . In that paper, it is proven that a goal is FP if and only if both the goal and its complement satisfy two properties called *monotonicity* and *selectivity*. On the other hand, a goal (but not its complement) being monotone and selective is not sufficient for HP. Moreover, HP on unitialized games has been specifically investigated by Kopczyński in [6, 7]. There, the author defines sufficient conditions for a goal to be HP on all finite arenas. However, no characterization of half-positional goals has been found so far. Positionality of games with infinitely many moves has been studied in [2, 5].

In his work, Kopczyński proves that if a goal is *concave* and *prefix-independent* then it is HP. Intuitively, a goal is concave if any (ordered) interleaving of two plays that are not in the goal results in a play that is not in the goal either. Moreover, a goal is prefix-independent if, by adding or removing any finite prefix to a play in the goal, the resulting play also belongs to the goal. In this paper, we investigate half-positionality on finite initialized arenas and provide a novel sufficient condition for a goal to be HP on all such arenas. We prove that if a goal is *strongly monotone* and *strongly concave*, then it is HP on initialized games. As the names suggest, strong monotonicity is derived from the notion of monotonicity of [4] and strong concavity refines the notion of concavity defined in [6]. We prove that our condition constitutes an improvement over that defined in [6], because it allows to characterize a broader set of goals as HP on initialized games. Several examples show that our condition is robust, in the sense that it is not trivial to further strengthen the result.

Overview. The rest of the paper is organized as follows. In Section 2, we present the formal framework. In Section 3, we introduce the new properties of goals sufficient to ensure half-positionality. We prove that such properties describe a wider set of goals than the properties in [6] and we show that some weaker conditions are not sufficient. In Section 4, we prove that our conditions are sufficient but not necessary to half-positionality. In Section 5, we analyze the conditions of [4], relating them to half-positionality. We show that natural stronger forms of such conditions are not sufficient, and we conclude by defining a characterization for half-positionality on game graph whose nodes belong all to one player only. Finally, we provide some conclusions in Section 6.

2 Preliminaries

Let X be a set and i be a positive integer. By X^i, we denote the Cartesian product of X with itself i times and by X^* (resp., X^ω) the set of finite (resp., infinite) sequences of elements of X. The set X^* also contains the *empty word* ε. A *language* (resp., ω-*language*) on the alphabet X is a subset of X^* (resp. X^ω).

We denote by $\mathbb{N}$ the set of non-negative integers. For a non-negative integer k, let $[k] = \{0, 1, \dots, k\}$. A *word* on the alphabet $[k]$ is a finite or infinite sequence of elements of $[k]$, a *language* over the alphabet $[k]$ is a set of words over $[k]$.

2.1 Colored Games

Finite automata. A *finite state automaton* is a tuple (X, Q, δ, q_0, F) where X is an *alphabet*, Q a finite set of *states*, $q_0 \in Q$ an *initial state*, $F \subseteq Q$ a set of *final states* and $\delta : Q \times X \to 2^Q$ a *transition function*. A *run* of the automaton on a sequence $x_1 \dots x_k \in X^*$, is a sequence $q_0 \dots q_k \in Q^*$ such that for each $i \in \{1, \dots, k\}$ we have $q_i \in \delta(q_{i-1}, x_i)$. A word $x \in X^*$ is said *accepted* by the automaton if there exists a run $q_0, \dots, q_k$ on x ending in a final state $q_k \in F$. A language is said to be *regular* iff there exists a finite state automaton that accepts all and only the words belonging to it.

Arenas. A *k-colored arena* is a tuple $A = (V_0, V_1, E)$, where V_0 and V_1 are a partition of a finite set V of *nodes*, and $E \subseteq V \times [k] \times V$ is a set of *colored edges* such that for each node $v \in V$ there is at least one edge exiting from v. A colored edge $e = (u, a, v) \in E$ represents a connection *colored* with a from the node u, named *source* of e, to the node v, named *destination* of e. In the following, we simply call a k-colored arena an *arena*, when k is clear from the context. For a node $v \in V$, we call

${}_vE = \{(v,a,w) \in E\}$ and $E_v = \{(w,a,v) \in E\}$ the sets of edges exiting and entering v, respectively.

For a color $a \in [k]$, we denote by $E(a) = \{(v,a,w) \in E\}$ the set of edges colored by a. A *finite path* ρ is a finite sequence of edges $(v_i, a_i, v_{i+1})_{i\in\{0,\dots,n-1\}}$, and its *length* $|\rho|$ is the number of edges it contains. We use $\rho(i)$ to indicate the i-th edge of ρ. Sometimes, we write the path ρ as $v_0v_1\dots v_n$, when the colors are not important. An *infinite path* is defined analogously, i.e., it is an infinite sequence of edges $(v_i, a_i, v_{i+1})_{i\in\mathbb{N}}$. For a (finite or infinite) path ρ and an integer i, we denote by $\rho^{\leq i}$ the *prefix* of ρ containing i edges. The *color sequence* of a finite (resp. infinite) path $\rho = (v_i, c_i, v_{i+1})_{i\in\{0,\dots,n-1\}}$ (resp. $\rho = (v_i, c_i, v_{i+1})_{i\in\mathbb{N}}$) is the sequence $Col(\rho) = (c_i)_{i\in\{0,\dots,n-1\}}$ (resp. $Col(\rho) = (c_i)_{i\in\mathbb{N}}$) of the colors of the edges of ρ. For two color sequences $x, y \in [k]^\omega$, the *shuffle* of x and y, denoted by $x \otimes y$ is the language of all the words $z_1z_2z_3\ldots \in [k]^\omega$, such that $z_1z_3\ldots z_{2h+1}\ldots = x$ and $z_2z_4\ldots z_{2h}\ldots = y$, where $z_i \in [k]^*$ for all $i \in \mathbb{N}$. For two languages $M, N \subseteq [k]^\omega$, the *shuffle* of M and N is the set $M \otimes N = \bigcup_{m\in M, n\in N} m \otimes n$.

Games. A *k-colored game* is a pair $G = (A, W)$, where $A = (V_0, V_1, E)$ is a k-colored arena, and $W \subseteq [k]^\omega$ is a set of color sequences called *goal*. By $\overline{W}$ we denote the set $[k]^\omega \setminus W$. An *initialized k-colored game* is a triple (A, W, v_{ini}) where (A, W) is a k-colored game and v_{ini} is a node of A. Informally, we assume that the game is played by two players, referred to as player 0 and player 1. Starting from a node v ($v = v_{ini}$ for an initialized game), the players construct a path on the arena A; such a path is called *play*. Once the partial play reaches a node $v \in V_0$, player 0 chooses an edge exiting from v and extends the play with this edge; once the partial play reaches a node $v \in V_1$, player 1 makes a similar choice. Player 0's aim is to make the play to have color sequence in W, while player 1's aim is to make the play have color sequence in $\overline{W}$. We now define some notation in order to formalize the previous intuitive description.

For $h \in \{0, 1\}$, let $E_h = \{(v,c,w) \in E \mid w \in V_h\}$ be the set of edges ending into nodes of player h. Moreover, let ε be the empty word. A *strategy* for player h is a function $\sigma_h : V_h \cup (E^*E_h) \to E$ such that, if $\sigma_h(e_0 \dots e_n) = e_{n+1}$, then the destination of e_n is the source of e_{n+1}, and if $\sigma_h(v) = e$, then the source of e is v. Intuitively, σ_h fixes the choices of player h for the entire game, based on the previous choices of both players. The value $\sigma_h(v)$ is used to choose the first edge in the game when the starting node is v. A strategy σ_h is *positional* iff its choices depend only on the last node of the partial play, i.e., for all partial plays ρ with last node v, it holds that $\sigma_h(\rho) = \sigma_h(v)$. A play $(e_i)_{i\in\mathbb{N}} \in E^\omega$ with starting node v is *consistent* with a strategy σ_h iff *(i)* if $v \in V_h$ then $e_0 = \sigma_h(\varepsilon)$, and *(ii)* for all $i \in \mathbb{N}$, if $e_i \in E_h$ then $e_{i+1} = \sigma_h(e_0 \dots e_i)$.

An infinite play ρ is *winning* for player 0 (resp. player 1) iff $Col(\rho) \in W$ (resp. $Col(\rho) \notin W$). An infinite play that is not winning for a player is also called *losing* for that player. Note that, in a game $G = (A, W)$, given two strategies, σ for player 0 and τ for player 1, there exists only one play consistent with both of them and starting at a node v. This is due to the fact that the two strategies uniquely determine the next edge at every step of the play. We call such a play $P(\sigma, \tau, v)$. A strategy for player h is *winning* on $G = (A, W)$ from v iff all plays on A, starting at v and consistent with that strategy are winning for player h. The *winning set* of a strategy τ for player h

on a game G is the set $Win_h(G,\tau)$ of all and only the nodes v such that τ is winning from v. The *winning set* of player h on a game G is the set $Win_h(G)$, containing all and only the nodes v such that there exists a winning strategy for player h on G from v. A game G is *determined* iff $Win_0(G) \cup Win_1(G) = V$. A goal W is *determined* iff all games $G = (A,W)$ are determined.

A goal W is *half-positional* on an arena A from a node v iff player 0 has a positional strategy winning on (A,W) from v. A goal W is *half-positional* on an arena A iff player 0 has a positional strategy σ such that $Win_0(G,\sigma) = Win_0(G)$. A goal W is *half-positional* iff it is half-positional on all arenas A. A goal W is *half-positional* on initialized games iff it is half-positional on all arenas A from each node $v \in Win_0(A,W)$.

A goal W is *full positional* on an arena A from a node v iff both players have a positional strategy winning on (A,W) from v. A goal W is *full positional* on an arena A iff both players $h \in \{0,1\}$ have a positional strategy that is winning from all nodes $v \in Win_h(A,W)$. A goal W is *full positional* iff it is full positional on all arenas A.

2.2 A Sufficient Condition for Half Positionality

Half-positionality states that the winning capability of a player is not weakened if we force that player to only use memoryless strategies, i.e., to only base her decisions on the current game state. In [6], Kopczyński defines two properties of goals, that together are sufficient for half-positionality. The first one states that every time the play passes through a node controlled by player 0, this player always prefers progressing via the same edge, rather than alternating between different ones. This property is captured by the concept of *concavity*, which states that the shuffle of two losing plays results in a set of losing plays.

Definition 1 ***(Concavity)*** A goal $W \subseteq [k]^\omega$ is *concave* if, for each pair $x,y \notin W$, we have that $(x \otimes y) \cap W = \emptyset$.

Every time player 0 reaches a node with multiple exiting edges, he has a choice between many positional behaviors. If one of them is winning, player 0 uses that one. If they are all losing, concavity ensures that they cannot be alternated in order to give a non-memoryless winning behavior. However, it is still possible that the optimal positional choice for player 0 depends on some finite prefix up to that node. This case is ruled out by the prefix-independence property, which states that the winning value of a word does not change when we modify some finite prefix.

Definition 2 ***(Prefix-independence)*** A goal $W \subseteq [k]^\omega$ is *prefix-independent* if, for all infinite words $x \in [k]^\omega$ and for all finite words $z \in [k]^*$, we have that $x \in W$ if and only if $z \cdot x \in W$.

Together, the concavity and the prefix-independent properties constitute a sufficient condition not only to half-positionality, but also to determinacy.

Theorem 1 *[6] All concave and prefix-independent goals are determined and half-positional.*

2.3 Comparing Words and Languages

Given a goal $W \subseteq [k]^\omega$, we introduce the following order relations. For all words $x, y \in [k]^\omega$, we say that *(i)* x is *not better* than y (written $x \leq_W y$), when they are both losing or y is winning, and *(ii)* y is *better* than x (written $x <_W y$), when y is winning and x is losing. In the same way, for all ω-languages $M, N \in [k]^\omega$, we say that *(i)* M is not better than N, in symbols $M \leq_W N$, to mean that if M contains a winning word then N contains a winning word as well, and *(ii)* N is better than M, in symbols $M <_W N$, to mean that M contains only losing words and N contains at least a winning word. For ease of reading, when the goal W is clear from the context, we simply write $x \leq y$, $x < y$, $M \leq N$ and $M < N$, respectively. With the following two lemmas, we reformulate the definition of concavity and prefix-independence in terms of languages, rather than single words.

Lemma 1 *A goal $W \subseteq [k]^\omega$ is prefix-independent if and only if, for all color sequences $x \in [k]^*$ and sets of color sequences $M \subseteq [k]^\omega$, we have that $xM \leq M$ and $M \leq xM$.*

Proof Suppose that W is prefix-independent. If M contains a winning word m, then xM contains the winning word xm, and we have both $xM \leq M$ and $M \leq xM$. If M contains only losing words m, then xM contains only losing words xm and we have both $xM \leq M$ and $M \leq xM$.

Suppose now that, for all ω-languages $M \subseteq [k]^\omega$, we have $xM \leq M$ and $M \leq xM$. Moreover, suppose by contradiction that there exists a word $m \in W$ such that $xm \notin W$. For the language $M = \{m\}$, we do not have $M \leq xM$. A similar arguments applies when $m \in \overline{W}$ and $xm \in W$. □

Lemma 2 *A goal $W \subseteq [k]^\omega$ is concave if and only if, for all languages $M, N \subseteq [k]^\omega$, we have that $M \otimes N \leq M \cup N$.*

Proof Suppose that W is concave. For all $M, N \subseteq \overline{W}$, we have that $M \otimes N \subseteq \overline{W}$. So, for all languages $M, N \subseteq [k]^\omega$, if M or N contains a word in W, we have in both cases $M \otimes N \leq M \cup N$; conversely, if M and N contain only losing words, by hypothesis so does $M \otimes N$. Hence, we have that $M \otimes N \leq M \cup N$.

Suppose now that for all languages $M, N \subseteq [k]^\omega$ we have $M \otimes N \leq M \cup N$. Then, if M and N contain only losing words, $M \otimes N$ must contain only losing words too. Thus, for all $M, N \in \overline{W}$ we have that $M \otimes N \subseteq \overline{W}$. Hence, W is concave. □

2.4 A Characterization for Full Positionality

Before starting our investigation on a possible characterization for the property of half-positionality, it is useful to recall a similar result due to Gimbert and Zielonka about *full* positionality [4]. Such a result does not only allow us to prove the positionality of some goals, but it also gives us inspiration for a novel sufficient condition for half-positionality. We start with the definition of the infinite extension of a language of finite words.

Definition 3 **(*Infinite extension*)** Let $M \subseteq [k]^*$ be a language of finite words, then the *infinite extension* of M is the set $\langle M \rangle$ of infinite words $x \in [k]^\omega$ such that each prefix of x is a prefix of at least one word in M.

Next, we introduce the notion of selectivity. When player i uses a memoryless strategy, we can construct a subgraph that is obtained by removing from the nodes controlled by player i all the edges not used by that strategy. In such subgraph, there is only one exiting edge from every node of player i. Hence, player $1-i$ makes all the choices. Since the graph may be considered as the representation of a finite state automaton, the color sequences of all paths constructed by player $i-1$ form a regular language. Also, the set of all finite paths from a node of player $i-1$ to itself is a regular language. Hence, in order to obtain half-positionality for player i, we do not need to ask that player i does not prefer switching between different arbitrary complex behaviors, but just that she does not prefer switching between two different regular languages. This property is captured by the notion of selectivity. It states that, given three regular languages M, N, and K, switching infinitely often between M and N or switching finitely often between them and then progressing with a path in K is not better than always staying in the same language.

Definition 4 **(*Selectivity*)** A goal $W \subseteq [k]^\omega$ is *selective* if and only if for all $x \in [k]^*$ and for all regular languages $M, N, K \subseteq [k]^*$, we have that $x\langle (M \cup N)^* K \rangle \leq x\langle M^* \rangle \cup x\langle N^* \rangle \cup x\langle K \rangle$.

However, selectivity does not avoid that a player's choice in a given node may depend on the finite prefix up to that node. Prefix-independence can solve this problem. However, such a strong property is not needed. The following ordering between ω-languages paves the way for a weaker property that suffices for our purposes: monotonicity.

Definition 5 **(*Definitive order relation*)** Given a goal $W \subseteq [k]^\omega$ and two ω-languages $M, N \subseteq [k]^\omega$, we write $M \sqsubseteq N$ if for all words $x \in [k]^*$, it holds that $xM \leq xN$.

The relation $M \sqsubseteq N$ may be interpreted as "N is definitively not worse than M". When player i at a given point needs to choose between several behaviors, it is sufficient to take the one which is definitively not worse than the others. Indeed, no matter the prefix up to that point, that choice gives better results. Accordingly, the relation $\sqsubseteq$ needs to be total, otherwise a dominant behavior may not exist.

Definition 6 **(*Monotonicity*)** A goal $W \subseteq [k]^\omega$ is *monotone* if and only if for all pairs of regular languages $M, N \subseteq [k]^*$ it holds that $\langle M \rangle \sqsubseteq \langle N \rangle$ or $\langle N \rangle \sqsubseteq \langle M \rangle$.

Definition 6 is equivalent to the definition of monotonicity given in [4], which states that W is monotone if and only if for all words $x \in [k]^*$ and all regular languages $M, N \subseteq [k]^\omega$, if $x\langle M \rangle < x\langle N \rangle$ then for all words $y \in [k]^*$ it holds $y\langle M \rangle \leq y\langle N \rangle$. The equivalence is proved by the following lemma.

Lemma 3 *A goal $W \subseteq [k]^\omega$ is* monotone *if and only if, for all words $x \in [k]^*$ and all regular languages $M, N \subseteq [k]^\omega$, it holds that $x\langle M \rangle < x\langle N \rangle$ implies that for all $y \in [k]^*$ it is $y\langle M \rangle \leq y\langle N \rangle$.*

Proof [if] Suppose that the r.h.s. of the double implication holds. Assume then that $\langle M\rangle \not\sqsubseteq \langle N\rangle$. Hence, by definition there exists $x \in [k]^*$ such that $x\langle N\rangle < x\langle M\rangle$. By the r.h.s. of the double implication, for all $y \in [k]^*$ we have $y\langle N\rangle \leq y\langle M\rangle$. So, $\langle N\rangle \sqsubseteq \langle M\rangle$ and the goal is monotone.

[only if] Suppose that W is monotone, and let $x \in [k]^*$ be a word and $M, N \subseteq [k]^\omega$ be two regular languages such that $x\langle M\rangle < x\langle N\rangle$. Then, by definition $\langle N\rangle \not\sqsubseteq \langle M\rangle$. By monotonicity of W, we have $\langle M\rangle \sqsubseteq \langle N\rangle$, i.e., for all $y \in [k]^*$ it holds that $y\langle M\rangle \leq y\langle N\rangle$, thus proving the r.h.s. of the double implication. □

The two previous properties lead to a characterization of full-positionality. The following is just a corollary of the more general result from Gimbert and Zielonka, who state it in the more general framework of optimization games, i.e., games equipped with a preference relation on plays, rather than a goal.

Theorem 2 **([4])** *A goal $W \subseteq [k]^\omega$ is full-positional on unintialized games if and only if both W and $\overline{W}$ are selective and monotone.*

Based on the above theorem, it is tempting to conjecture that W is half-positional if and only if it is selective and monotone. However, this is not the case: Lemma 15 shows that there exists a monotone and selective goal which is not half-positional. The next section presents a novel sufficient condition for half-positionality.

3 Strong Monotonicity and Strong Concavity

Concavity and prefix-independence are sufficient but not necessary for half-positionality. On the other hand, Theorem 2 provides a characterization for full-positionality by relaxing concavity and prefix-independence to selectivity and monotonicity, and applying them to both the goal and its complement. Unfortunately, selectivity and monotonicity of the goal are not sufficient for it to be half positional, as we will formally show in Section 5. So, it is reasonable to ask whether we can relax concavity and prefix-independence to some other properties, still sufficient for half-positionality. Our investigation starts from a counter-example showing that a full-positional goal does not need to be prefix-independent.

Lemma 4 *There exists a full-positional goal which is not prefix-independent*

Proof Consider the goal $W = 0(1^*0)^\omega \cup 1[1]^\omega$. It contains all and only the words x such that either x starts with color 1, or x starts with color 0 and contains infinitely many 0's. The goal is not prefix-independent: if $y \in [1]^\omega$ does not contain infinitely many 0's then $0 \cdot y$ is losing and $1 \cdot y$ is winning. However, the goal is half-positional, and in particular full-positional, as we show by applying Theorem 2. Precisely, we show that both W and $\overline{W} = [1]^\omega \setminus W = 0[1]^*1^\omega$ are monotone and selective.

1. W is selective. Consider $M, N, K \subseteq [1]^*$ and $x \in [1]^*$. Suppose that the set $x\langle M^*\rangle \cup x\langle N^*\rangle \cup x\langle K\rangle$ contains only losing words. Then there are two possible situations: *(i)* x is not empty and starts with color 0, or *(ii)* x is empty and all words in M, N, K start with color 0. In both cases, no word in M or N contains occurrences of

color 0, and $\langle K \rangle$ does not contain words with infinitely many 0's. Hence, $x\langle (M \cup N)^* K \rangle$ contains only words starting with color 0 and containing finitely many occurrences of 0. Thus, $x\langle (M \cup N)^* K \rangle$ is not better than $x\langle M^* \rangle \cup x\langle N^* \rangle \cup x\langle K \rangle$.

2. $\overline{W}$ is selective. Consider $M, N, K \subseteq [1]^*$ and $x \in [1]^*$. Suppose that the set $x\langle M^* \rangle \cup x\langle N^* \rangle \cup x\langle K \rangle$ contains only losing words w.r.t. $\overline{W}$ (i.e., it is a subset of W). First, suppose that x starts with color 0. Then, all words in M and N contain at least one 0, and all words in $\langle K \rangle$ contain infinitely many 0's. So, every word in the set $x\langle (M \cup N)^* K \rangle$ starts with 0 and contains infinitely many zeros and is losing with respect to $\overline{W}$. A similar argument applies when x is empty and there exists at least one word in M, one in N and one in $\langle K \rangle$ starting with 0. If, instead, x is not empty and starts with 1, or x is empty and all words in M, N, and $\langle K \rangle$ start with 1, the result is obvious.
3. W is monotone. Consider $M, N \subseteq [1]^*$ and $x \in [1]^*$ such that $x\langle M \rangle <_W x\langle N \rangle$. Then x is empty or it starts with color 0, or else both sets would be winning. Assume for simplicity that x is not empty, as the opposite case can be treated similarly. We have that $\langle M \rangle$ contains only words with finitely many occurrences of 0 and $\langle N \rangle$ contains a word with infinitely many occurrences of 0. Consider a new word $y \in [1]^*$; if y starts with color 0, we still have $y\langle M \rangle <_W y\langle N \rangle$. If y starts with color 1, both sets $y\langle M \rangle, y\langle N \rangle$ are winning. So, in both cases $y\langle M \rangle \leq_W y\langle N \rangle$.
4. $\overline{W}$ is monotone. Consider $M, N \subseteq [1]^*$ and $x \in [1]^*$ such that $x\langle M \rangle <_{\overline{W}} x\langle N \rangle$. Then, x is empty or it starts with color 0, or else both sets would be losing. Once again, assume for simplicity that x is not empty. We have that $\langle M \rangle$ contains only words with infinitely many 0's and $\langle N \rangle$ contains a word with finitely many 0's. Consider another word $y \in [1]^*$; if y starts with color 0 we still have $y\langle M \rangle <_{\overline{W}} y\langle N \rangle$. If y starts with color 1, both sets $y\langle M \rangle, y\langle N \rangle$ are losing. So, in both cases $y\langle M \rangle \leq_{\overline{W}} y\langle N \rangle$. □

This counter-example shows that, as it has been discussed in the introduction to monotonicity, prefix independence is a strong property that is not needed for positionality. Indeed, even if the winning nature of the decision of a player on a node depends on the prefix up to that node, it is still possible that a decision performs better than the others on all prefixes. Then, like in the monotonicity property, we need to ask that given two decisions for a given player, there do not exist two prefixes that completely change the winning nature of the two decisions. Since the behavior of a player is no longer ensured to be regular by the use of a positional strategy by the other player, we make use of a strong version of monotonicity that takes into account also non-regular behaviors.

Definition 7 (***Strong monotonicity***) A goal $W \subseteq [k]^\omega$ is *strongly monotone* if and only if, for all pairs of ω-languages $M, N \subseteq [k]^\omega$, it holds that $M \sqsubseteq N$ or $N \sqsubseteq M$.

Like monotonicity, strong monotonicity can be defined equivalently as follows.

Lemma 5 *A goal $W \subseteq [k]^\omega$ is strongly monotone if and only if, for all words $x \in [k]^*$ and languages $M, N \subseteq [k]^\omega$, it holds that $xM < xN$ implies that for all $y \in [k]^*$ it is $yM \leq yN$.*

As one may easily guess, strong monotonicity is a weaker property compared to prefix-independence, as shown by the following lemma.

Lemma 6 *All prefix-independent goals are strongly monotone. Moreover, there is a goal which is strongly monotone, but not prefix-independent.*

Proof For the first part, we have by hypothesis that, for all $x \in [k]^*$, and $M \subseteq [k]^\omega$, it holds that $M \leq xM \leq M$. Now, take two languages $M, N \subseteq [k]^\omega$, and suppose that there exists an $x \in [k]^*$ such that $xM < xN$, then for all $y \in [k]^*$ we have $yM \leq M \leq xM \leq xN \leq N \leq yN$.

For the second part, let $k = 1$, a strongly monotone and prefix-dependent goal is given by the language of all words containing at least one 0, i.e., $W = [k]^*0[k]^\omega$. It is easy to see that the goal is not prefix-independent, because the word 1^ω is losing while the word 01^ω is winning. We show that W is strongly monotone. Consider two languages $M, N \subseteq [k]^\omega$, and suppose that there exists an $x \in [k]^*$ such that $xM < xN$, then xN contains a winning word and xM contains only losing words. Observe first that x cannot contain 0, or else all words in xM would be winning. So $x \in 1^*$, there exists a word in N that contains 0, and all words in M contain only 1's. So, for each $y \in [k]^*$, there is always a word in yN containing 0. Since yN contains a winning word, we have $yM \leq yN$. □

Strong monotonicity seems a good substitute to prefix-independence. Unfortunately, the following lemma shows that strong monotonicity cannot replace prefix-independence in the hypotheses of Theorem 1.

Lemma 7 *There is a strongly monotone, concave and non-half-positional goal.*

Proof For $k = 1$, the strongly monotone and concave goal is $W = [k]^*01^\omega$. We prove first that the goal is strongly monotone and concave. A word is losing if and only if it is either 1^ω or it does not have 1^ω as a suffix. Let $x \in [k]^*$, $n, m \in [k]^\omega$ with $xn, xm \notin W$. We distinguish two cases. First, assume that x does not contain 0. Then, n and m may be both 1^ω in which case $x(m \otimes n) = 1^\omega$ or at least one between n and m contains 0 infinitely often, thus the shuffle of n and m contains only words that pick colors from both the sequences infinitely often and thus only words that contain 0 infinitely often. So, $x(m \otimes n)$ contains a losing word even in this case. Instead, assume that x contains 0. Then, n and m contain 0 infinitely often and the same reasoning above applies. So the goal is concave.

Let $x \in [k]^*$, $n, m \in [k]^\omega$ such that $xm \notin W$ and $xn \in W$. We prove strong monotonicity by showing that for all $y \in [k]^*$ it holds that $ym \notin W$ or $yn \in W$. We again distinguish two cases. First, assume that x does not contain a 0. Then, n contains 0 and a suffix 1^ω. Thus, for every $y \in [k]^*$, we have $yn \in W$ since it contains 0 and a suffix 1^ω. Instead, assume that x contains a 0. In this case, m contains 0 infinitely often, and for every $y \in [k]^*$ we have $ym \notin W$ since ym contains 0 infinitely often. The above goal is not half-positional in the arena $(\{v\}, \emptyset, \{(v,0,v), (v,1,v)\})$, depicted in Fig. 1(a). In such arena, player 0 wins by choosing at least once the edge with color 0 and then always the edge with color 1. □

Observe that, in the previous counterexample, the key element that does not allow half positionality is the fact that player 0 prefers switching between two different behaviors finitely often and then progressing indefinitely along one of them. However,

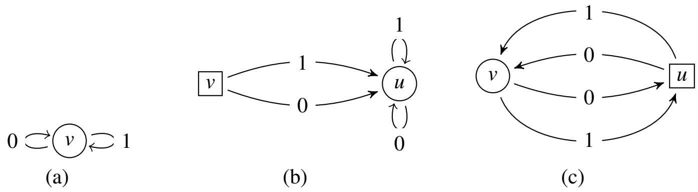

Fig. 1 Three game arenas.

concavity just requires that player 0 prefers following a fixed behavior rather than switching between two different ones *infinitely often*. Thus, we introduce a modification to the property of concavity, requiring not only that alternating infinitely often between two losing words yields a losing word, but also that alternating *finitely* often between two losing words and then progressing along one of them yields a losing word.

Definition 8 **(*Strong shuffle*)** For two color sequences $x, y \in [k]^\omega$, the *strong shuffle* of x and y, denoted by $x \otimes_s y$, is the language containing

1. the set $x \otimes y$;
2. the words $z_1 z_2 \dots z_l z' \in [k]^\omega$, for odd l, $z_i \in [k]^*$ and $z' \in [k]^\omega$, such that it holds $x = z_1 z_3 \dots z_l z'$ and $y = z_2 z_4 \dots z_{l-1} y'$, for some $y' \in [k]^\omega$;
3. the words $z_1 z_2 \dots z_l z' \in [k]^\omega$, for even l, $z_i \in [k]^*$ and $z' \in [k]^\omega$, such that it holds $x = z_1 z_3 \dots z_{l-1} x'$ and $y = z_2 z_4 \dots z_l z'$, for some $x' \in [k]^\omega$.

For two languages $M, N \subseteq [k]^\omega$, the *strong shuffle* of M and N is the set $M \otimes_s N = \cup_{n \in N, m \in M} (m \otimes_s n)$.

Definition 9 **(*Strong concavity*)** A goal $W \subseteq [k]^\omega$ is *strongly concave* if and only if, for all words $x \in [k]^*$, $n, m \in [k]^\omega$, and $z \in x(m \otimes_s n)$, it holds that if $z \in W$ then either $xn \in W$ or $xm \in W$.

It is immediate to see that a strongly concave goal is concave too. In the following, we make use of an equivalent definition of strong concavity that operates on languages.

Lemma 8 *A goal $W \subseteq [k]^\omega$ is strongly concave if and only if, for all languages $M, N \subseteq [k]^\omega$, it holds that $(M \otimes_s N) \sqsubseteq M \cup N$.*

Proof [if] Suppose that for all languages $M, N \subseteq [k]^\omega$, it holds that $(M \otimes_s N) \sqsubseteq M \cup N$. In particular for $m, n \in [k]^\omega$ let $M = \{m\}$ and $N = \{n\}$. We obtain $m \otimes_s n \sqsubseteq \{m, n\}$, and hence W is strongly concave.

[only if] Suppose that W is strongly concave. Consider two languages $M, N \subseteq [k]^\omega$ and suppose by contradiction that $M \otimes_s N \not\sqsubseteq M \cup N$. Then there exists a word $x \in [k]^*$ such that all words in $xM \cup xN$ are losing and there exists a winning word $z \in x \cdot (M \otimes_s N)$. By definition of shuffle, there exist two losing words $m \in M$ and $n \in N$ such that $z \in m \otimes_s n$, which contradicts the strong concavity property. □

Even the property of strong concavity is not sufficient to ensure half positionality.

Lemma 9 *There is a strongly concave goal which is not half-positional.*

Proof For $k = 1$, the strongly concave goal is $W = 0^\omega \cup 1^\omega$. Two losing words n and m contain at least an occurrence of color 1 and an occurrence of color 0, thus every word in their strong shuffle will contain at least an occurrence of color 1 and an occurrence of color 0 and it will be losing. So the strong concavity of the goal is proved. The above goal is not half-positional in the 2-colored arena $(\{u\}, \{v\}, \{(v,0,u),(v,1,u),(u,0,u),(u,1,u)\})$, showed in Figure 1(b). In this arena, if the game starts in node v, player 0 wins by choosing forever the edge $(u,0,u)$ or the edge $(u,1,u)$, depending on what color was chosen by player 1 when moving from v to u. □

In the previous counterexample, by choosing a different prefix, player 1 can exchange the winning nature of the following choices of player 0. That is why strong monotonicity is essential since it somehow allows player 0 to operate while forgetting the past decisions taken by player 1.

We argue now that the two introduced properties of strong monotonicity and strong concavity are strictly less restrictive than the properties of prefix independence and concavity.

Lemma 10 *All concave and prefix-independent winning conditions are also strongly monotone and strongly concave.*

Proof By Lemma 6 we already have that a prefix-independent goal is strongly monotone. It remains to show that a concave and prefix-independent goal is strongly concave.

For a language $M \subseteq [k]^\omega$, let $\mathit{suff}(M)$ and $\mathit{pref}(M)$ be the sets of suffixes and prefixes of words in M, respectively. By concavity, for all $M,N \subseteq [k]^\omega$ we have $M \otimes N \leq M \cup N$ and by prefix independence we have for all $M \in [k]^\omega$ and for all $x \in [k]^*$ $M \leq xM \leq M$. Take any word $x \in [k]^*$, and any two languages $M,N \subseteq [k]^\omega$. Then we have $x(M \otimes_s N) = x(M \otimes N) \cup x \cdot \mathit{pref}(M \otimes N) \cdot \mathit{suff}(N) \cup x \cdot \mathit{pref}(M \otimes N) \cdot \mathit{suff}(M)$. First by prefix independence and then by concavity we have $x(M \otimes N) \leq M \otimes N \leq M \cup N \leq x(M \cup N) = xM \cup xN$. Then, $x \cdot \mathit{pref}(M \otimes N) \cdot \mathit{suff}(T) \leq \mathit{suff}(T) \leq xT \leq xM \cup xN$, where $T \in \{M,N\}$. So, we have $x(M \otimes_s N) \leq xM \cup xN$. □

Lemma 11 *There exists a strongly monotone and strongly concave goal which is not prefix independent.*

Proof Let $k = 1$, the goal is given by the set of words that either start with 1, or start with 0 and contain infinitely many 0's, i.e., $W = 0(1^*0)^\omega \cup 1[k]^*$. It is easy to see that the goal is not prefix-independent: indeed, for $M = 1^\omega$ we have that $0 \cdot M \leq M$, but $M \not\leq 0 \cdot M$, since M contains only winning words and $0 \cdot M$ only losing ones.

Next, we prove that the goal is strongly monotone. Consider $M,N \subseteq [k]^*$ and $x \in [k]^*$ and suppose that $xM < xN$, so that xN contains a winning word and xM contains only losing ones. Observe that x does not start with 1, otherwise all words in xM would be winning. We distinguish two cases: $x = \varepsilon$ and x starts with 0. If $x = \varepsilon$ then all words in M start with 0 and have a suffix equal to 1^ω. Now for all $y \in 1[k]^*$ we have $yM \leq yN$ since all the words in both languages are winning; for all $y \in 0[k]^*$

we have $yM \leq yN$ because all the words in yM are losing since they start with 0 and have a suffix 1^ω. If instead x starts with 0 then there exists a word $n \in N$ that contains infinitely many 0's. For every $y \in [k]^*$ the word yn contains infinitely many 0's and it is winning; thus for all $y \in [k]^*$ we have $yM \leq yN$.

Finally, we prove that the goal is strongly concave. Consider $x \in [k]^*$, $M,N \subseteq [k]^\omega$ and $K \subseteq [k]^*$. We want to prove that $x(M \otimes_s N) \leq xM \cup xN$. If the r.h.s. of the inequality contains a winning word, the inequality trivially holds. So, suppose that the r.h.s. does not contain a winning word, so it cannot be $x \in 1[k]^*$ but it must be $x \in 0[k]^* \cup \{\varepsilon\}$. If x starts with 0, every word in M,N contains a suffix 1^ω and all words in $M \otimes_s N$ contain a suffix 1^ω. So, $M \otimes_s N$ contains only losing words. If $x = \varepsilon$, every word in M,N contains a suffix 1^ω and starts with 0, so all words in $M \otimes_s N$ contain a suffix 1^ω and start with 0, and therefore they are losing. □

4 A Novel Sufficient Condition for Half-Positionality

In this section, we prove that determinacy, strong monotonicity and strong concavity are sufficient but not necessary conditions to half positionality on initialized games for player 0.

Theorem 3 *All determined, strongly monotone and strongly concave goals are half-positional on initialized games.*

Proof Let $W \subseteq [k]^\omega$ be a determined, strongly monotone and strongly concave goal, and let $A = (V_0, V_1, E)$ be a k-colored arena and v_{ini} be a starting node. The proof proceeds by induction on the number of edges exiting from the nodes controlled by player 0 in A. As the base case, assume that for each node controlled by player 0 there exists only one exiting edge. Then, player 0 has only one possible strategy, which is positional. So, the result is trivially true. Next, suppose that in the arena A there are n edges exiting from nodes of player 0 and that, for all arenas with at most $n-1$ edges exiting from nodes of player 0, player 0 has a positional strategy winning from v_{ini}. Let $t \in V_0$ be such that there is more than one edge exiting from t. We can partition the set of edges exiting from t in two disjoint non-empty sets E_α and E_β. Let A_α and A_β be the two arenas obtained from A by removing the edges of E_β and E_α, respectively. First, suppose that in G_α or G_β player 0 has a winning strategy. Then, by inductive hypothesis he has a positional winning strategy. It is easy to see that such a strategy is winning in G too. Indeed, since player 0 controls the node t, he is able to force the play to stay always in G_α or G_β. Suppose now that player 0 has no winning strategy in G_α and in G_β. We prove the thesis by showing that player 0 has no winning strategy in G. By determinacy, there exist two strategies τ_α and τ_β winning for player 1 in G_α and G_β, respectively.

Let σ be a strategy of player 0 in A, we show that there exists a strategy of player 1 in A winning against σ from v_{ini}. Observe that the two strategies τ_α and τ_β may not be able to play against σ on A, since they are not defined on some paths of A. If one of the plays $P(\sigma, \tau_\alpha, v_{\text{ini}})$, $P(\sigma, \tau_\beta, v_{\text{ini}})$ is well defined, then that play is in A_α or A_β, respectively, and so it is winning for player 1 who is using his winning strategy on that arena.

Suppose now that neither play is well defined; this happens only if by making σ play against τ_α (resp., τ_β) the node t is eventually reached and σ chooses an edge in A_β (resp. A_α). Let ρ_α (resp., ρ_β) be the finite play that starts in v_{ini}, ends in t and is consistent with σ and τ_α (resp., τ_β). Moreover, let x_α and x_β be respectively the color sequences of ρ_α and ρ_β. Let M_α (resp., M_β) be the set of color sequences of the plays consistent with τ_α (resp., τ_β) after ρ_α (resp., ρ_β). Formally, $M_\gamma = \{y \in [k]^\omega \mid \exists \pi \in E^\omega, \exists \sigma : P(\sigma, \tau_\gamma, v_{\text{ini}}) = \rho_\gamma \cdot \pi \text{ and } Col(\pi) = y\}$. Observe that $x_\alpha M_\alpha$ and $x_\beta M_\beta$ contain color sequences of plays consistent respectively with τ_α in A_α and τ_β in A_β, and such sequences are losing for player 0. We prove now that either $x_\alpha M_\beta$ or $x_\beta M_\alpha$ contains only losing words for player 0. Indeed, if $x_\alpha M_\beta$ contains a winning word, we have that $x_\alpha M_\alpha < x_\alpha M_\beta$. Then, by strong monotonicity we have that, for all $y \in [k]^*$, it holds $yM_\alpha \leq yM_\beta$ and in particular $x_\beta M_\alpha \leq x_\beta M_\beta$. Since $x_\beta M_\beta$ contains only losing words, so does $x_\beta M_\alpha$.

Suppose without loss of generality that $x_\beta M_\alpha$ contains only losing words. Then, we construct the strategy τ'_α, which behaves like τ_α on all partial plays which do not have a prefix ρ_β. When the partial play has a prefix ρ_β, it behaves like τ_α when it "sees" ρ_α in place of ρ_β. More formally, $\tau'_\alpha(\rho_\beta\pi) = \tau_\alpha(\rho_\alpha\pi)$, and in the other cases $\tau'_\alpha(\pi) = \tau_\alpha(\pi)$. Let $\tau'_\beta = \tau_\beta$.

Next, we construct another strategy τ for player 1 in A: at the beginning τ behaves like τ_β; when the play passes through t, depending on what subgraph the last edge from t chosen by player 0 belongs to, the strategy τ behaves like τ'_α or τ'_β, when they are applied only to the initial prefix up to t and all the loops from t to t, whose first edge belongs to A_α or A_β, respectively. Formally, for all prefixes π that do not contain t, we set $\tau(\pi) = \tau_\beta(\pi)$; for all prefixes π that do contain t, we can split π into three segments: *(i)* the prefix π' up to the first occurrence of t; *(ii)* a sequence of loops $\pi_{1,\gamma_1}, \ldots, \pi_{n,\gamma_n}$ from t to t, where $\gamma_i \in \{\alpha, \beta\}$ and the first edge of π_{i,γ_i} belongs to E_{γ_i}; *(iii)* the path π_γ from the last occurrence of t to the end of π, where $\gamma \in \{\alpha, \beta\}$ and the first edge of π_γ belongs to E_γ. We then set $\tau'(\pi) = \tau'_\gamma\big(\pi' \cdot (\prod_{\gamma_i = \gamma} \pi_{i,\gamma_i}) \cdot \pi_\gamma\big)$. The play $P(\sigma, \tau, v_{\text{ini}})$ coincides with $P(\sigma, \tau_\beta, v_{\text{ini}})$ up to t, so it starts with the finite path ρ_β, whose color sequence is x_β. After that prefix, the play alternates pieces of two plays: one in A_β consistent with $\tau'_\beta = \tau_\beta$, and the other in A_α consistent with τ'_α. So, the color sequence of the two suffixes are respectively in M_β and in M_α.[1] Hence, the color sequence of the suffix after x_β of the play $P(\sigma, \tau, v_{\text{ini}})$ lies in the shuffle of M_α and M_β. By strong concavity we have that $Col(P(\sigma, \tau, v_{\text{ini}})) \in x_\beta(M_\alpha \otimes_s M_\beta) \leq x_\beta M_\alpha \cup x_\beta M_\beta$. Since both $x_\beta M_\alpha$ and $x_\beta M_\beta$ contain only losing words, we have that $Col(P(\sigma, \tau, v_{\text{ini}}))$ is a losing word for player 0. Hence, for all strategies σ of player 0 there exists a strategy τ of player 1 winning over 0. We conclude that player 0 has no winning strategy. □

Since strong concavity implies concavity, the following result states that the conditions appearing as the hypotheses of the previous theorem and of Theorem 1 are not a complete characterization for half positional goals.

Lemma 12 *There exists a half-positional goal which is not full-positional nor concave.*

[1] Note that it is possible that one of the two suffixes does not progress indefinitely.

Proof The goal is $W = [2]^*0^\omega \cup [2]^*1^\omega \cup 2[2]^*2[2]^\omega$. It contains all the words that *(i)* definitively contain only occurrences of color 0, or *(ii)* definitively contain only occurrences of color 1, or *(iii)* start with color 2 and contain two occurrences of color 2.

First, the goal W is not concave. Indeed, for the losing word $m = 2 \cdot (10)^\omega$, in the shuffle set $m \otimes m$ there is the winning word $2 \cdot 2 \cdot (10)^\omega$.

The complementary goal is not selective. Indeed, consider $x = \varepsilon$, $M = N = K = 2(01)^*$. The set $x\langle M^*\rangle \cup x\langle N^*\rangle \cup x\langle K\rangle$ contains only losing words for the complementary goal, but the set $x\langle (M \cup N)^* K\rangle$ contains the winning word $22(10)^*$. By Theorem 2, the goal W is not full-positional.

The goal is monotone. Indeed, consider $M, N \subseteq [2]^\omega$ and $x \in [2]^*$ such that $xM < xN$. If xN contains a winning word with suffix 0^ω or 1^ω then, for all $y \in [2]^*$, the set yN contains a winning word and monotonicity holds. Otherwise, xN contains a winning word that starts with color 2 and contains at least a second occurrence of color 2. Since xM contains only losing words it contains no words with suffix 0^ω or 1^ω. Hence, yM may contain a winning word only if y starts with color 2. If $x = \varepsilon$, then adding a prefix y starting with color 2 to the word in N starting with color 2 does not change the winning nature of the word, hence $yM \leq yN$; If instead $x \neq \varepsilon$, then x starts with color 2, moreover x does not contain a second occurrence of color 2, else xM contains all winning words, and the words in M do not contain an occurrence of color 2. So yM can contain a winning word only if y contains two occurrence of color 2, but that makes yN contain only winning words, hence $yM \leq yN$.

The goal is half-positional. The proof of the statement is composed by the following steps on an arena $A = (V_0, V_1, E)$: *(i)* we prove that the goal $W' = [2]^*0^\omega \cup [2]^*1^\omega$ is half positional for player 0 and we determine the winning set U'_0 for player 0 on A, *(ii)* we prove that the goal $W'' = [2]^*0^\omega \cup [2]^*1^\omega \cup [2]^*2[2]^\omega$ is half-positional for player 0 and we determine the winning set U''_1 for player 1 on A, and *(iii)* since W is winning from U'_0 and losing from U''_1, it remains to determine the winning nature of the remaining nodes. We complete this last point by case analysis on the remaining nodes, and by determining a memoryless winning strategy for player 0 based on the winning strategy from step *(i)*.

[*(i)*] $W' = [2]^*0^\omega \cup [2]^*1^\omega$ is half-positional. The goal is prefix independent because the winning nature of a word depends only on an infinite suffix. The goal is concave, because given two words which do not have an infinite suffix with only color 0 or only color 1, then their shuffle cannot have such a suffix. Hence, by Theorem 1 the goal is half-positional for player 0. Let $U'_i = Win_i(A, W')$ and let σ' be a memoryless strategy for player 0 that is winning from all the nodes in U'_0. Observe that due to the prefix-independence, σ' traps the play in the nodes of U'_0, otherwise, as soon as the play is in U'_1 player 1 knows how to avoid a suffix with all colors 1 or all colors 0. Hence, the strategy σ' is only meaningful on the nodes in U'_0 and can be arbitrarily modified in all the other nodes, without loss of winning power in U'_0.

[*(ii)*] The goal $W'' = [2]^*0^\omega \cup [2]^*1^\omega \cup [2]^*2[2]^\omega$ is half-positional by Lemma 3 because it is strongly monotone, strongly concave and determined, as can be easily verified. Let $U''_i = Win_i(A, W'')$. Since $W' \subseteq W''$ we have $U'_0 \subseteq U''_0$ and $U''_1 \subseteq U'_1$. Observe that player 1 has no way to force the play from a node v in U''_0 to a node U''_1 unless the play has already seen an occurrence of color 2, otherwise after the play is in U'_1 player 1 can avoid any occurrence of color 2 and the suffixes 1^ω and 0^ω, which means that $v \in U''_1$.

[*(iii)*] We already know the that player 0 can win with respect to W on U'_0 by using the strategy σ', since $W' \subseteq W$. Moreover, player 0 cannot win from the nodes in U''_1 since $W \subseteq W''$. So let $U_? = (V_0 \cup V_1) \setminus (U'_0 \cup U''_1)$ be the set of nodes whose winning nature is still unknown. We define two increasing sequences $(M_i)_i$ and $(N_i)_i$ of sets of nodes in $U_?$, winning respectively for player 0 and player 1 for the goal $W''' = [2]^*0^\omega \cup [2]^*1^\omega \cup [2]^*2[2]^*2[2]^\omega$. We also construct a memoryless strategy σ which behaves like σ' on U'_0 and is winning for W''' on $U'_0 \cup (\cup_{i=0}^{+\infty} M_i)$.

1. N_0 contains all and only the nodes $v \in U_?$ such that *(i)* $v \in V_1$ and there is an edge from v to U''_1 or *(ii)* $v \in V_0$ and all the edges from v go to U''_1. Such node is winning for player 1 w.r.t. W''', since he can avoid a suffix 0^ω or 1^ω and a second occurrence of color 2. Observe that the edge from v to U''_1 is colored with 2, otherwise v would belong to U''_1.
2. For $i > 0$, N_i contains N_{i-1} and all the nodes $v \in U_?$ such that *(i)* $v \in V_1$ and there is an edge from v to N_{i-1} that is not colored with 2, or *(ii)* $v \in V_0$ and all the edges from v go into N_{i-1} and are not colored with 2. We have again that v is a winning node for player 1 w.r.t. W''', since it can directly move with an edge non-colored with 2 into a node $w \in N_{i-1}$ from which player 1 can force a path with at most one occurrence of color 2 and no suffix 0^ω or 1^ω.
3. M_0 is the set of all and only the nodes $v \in U_?$ such that *(i)* $v \in V_0$ and there is an edge from v to U''_0 colored with 2, or *(ii)* $v \in V_1$ and all the edges from v go to $U'_0 \cup U''_0$ and those that go to U''_0 are colored with 2. Then v is a winning node for player 0 w.r.t. W''', since it can move either into U'_0, from which player 0 can force a suffix 0^ω or 1^ω, or into a node $w \in U''_0$ via an edge colored with 2, and from w player 0 can force a play with another occurrence of color 2 or a suffix 0^ω or 1^ω. If $v \in V_0$ the winning memoryless strategy σ of player 0 is the one that chooses any edge colored with 2 and going into U''_0.
4. For $i > 0$, M_i is the set containing M_{i-1} and all the nodes $v \in U_?$ such that *(i)* $v \in V_0$ and there is an edge from v to M_{i-1}, or *(ii)* $v \in V_1$ and all edges from v go into M_{i-1} or into U'_0. Then v is winning for player 0, since from v we can move into a node $w \in M_{i-1}$ from which player 0 can force a path with two occurrence of color 2 or a suffix 0^ω or 1^ω. If $v \in V_0$ the winning memoryless strategy σ of player 0 is the one that chooses any edge going into M_{i-1}.

Since $U_?$ is finite, the increasing sequences M_i and N_i converge to $M_\infty = \cup_{i=0}^{\infty} M_i$ and $N_\infty = \cup_{i=0}^{\infty} N_i$ within a finite number of steps. So, N_∞ is losing for player 0 for W''' and, hence, it is losing for player 0 also for the goal W. On the other hand, among the nodes of M_∞, only those who can start with color 2 are winning for player 0 and the others are losing, because those nodes are in $U_?$ and from them player 0 cannot force a suffix with colors 0^ω or 1^ω or path with 2 as first color. Hence, the only nodes $v \in M_\infty$ winning for player 0 are those such that *(i)* $v \in V_1$ and all edges exiting from v, that do not go into U'_0, are colored with color 2 or *(ii)* $v \in V_0$ and there is an exiting edge colored with color 2.

At this point there is still one set of nodes left, namely, $U_{out} = U_? \setminus (M_\infty \cup N_\infty)$. The only possibility is that for each $v \in U_{out}$ either *(i)* $v \in V_1$ and there exists an edge non-colored with color 2 going into U_{out}, and all edges from v either go into U_{out} or go into $U'_0 \cup M_\infty$, or *(ii)* $v \in V_0$ and all the edges from v go in U_{out} and are not colored with 2. Hence, player 1 has a way to force paths in U_{out} that never passes through color 2. However, all such paths must have a suffix in 0^ω or 1^ω otherwise v would belong to U''_1. Hence, it does not matter what player 0 does in U_{out} because if player 1 does not come out, player 1 loses. So, player 1 can win only by moving out of U_{out}, (it can otherwise $U_{out} \subseteq U'_1$). If the play started with color 2, then player 1 loses because when he moves from U_{out}, the play goes into $U'_0 \cup M_\infty$, where player 0 can force another occurrence of color 2 or a suffix in 0^ω or 1^ω. If the play does not start with color 2, player 1 wins because in general in $U_?$, player 1 can force a path with no suffix 0^ω or 1^ω. So, in the winning set of player 0 we put also the nodes $v \in U_{out}$ such that *(i)* $v \in V_1$ and all edges exiting from v, that do not go into U'_0, are colored with color 2 or *(ii)* $v \in V_0$ and there is an exiting edge colored with color 2. □

Also, not all full-positional goals are concave.

Lemma 13 *There exists a goal that is full positional but not concave.*

Proof Let $k = 1$, the full positional goal is $W = [k]^*1[k]^*1[k]^\omega$. The goal states that player 0 tries to make color 1 occur at least twice. We show that the goal is not concave: let $x = \varepsilon$, $n, m = 10^\omega$, then we have $xn, xm \notin W$, but $t = 110^\omega \in m \otimes n$ with $xt \in W$, hence the goal is not concave. Intuitively, the goal is positional for player 0 because in every point in a play player 0 does not need to remember the past, but just tries to form a path that passes through as many edges colored with 1 as possible. We prove that the goal is full-positional through Theorem 2. Precisely, we prove that both W and $\overline{W} = [1]^* \setminus W$ are selective and monotone. Observe that $\overline{W}$ is the set of all the words containing at most one occurrence of color 1.

1. W is selective. Suppose by contradiction that W is not selective. Then, there exist $x \in [k]^*$ and $M, N, K \subseteq [k]^*$ such that $x\langle(M \cup N)^*K\rangle = x\langle(M \cup N)^*\rangle \cup x(M \cup N)^*\langle K\rangle$ contains a winning word and $x\langle M^*\rangle \cup x\langle N^*\rangle \cup x\langle K\rangle$ contains only losing words. Observe that no word in M or N contains 1, or else if $m \in M \cup N$ contains a 1, $xm^\omega \in x\langle M^*\rangle \cup x\langle N^*\rangle$ contains infinitely many 1's and it is a winning word. So, the words in the set $x\langle(M \cup N)^*\rangle$ do not contain 1 and they are losing. Moreover, since $x\langle K\rangle$ does not contain more than one 1, the words in $x(M \cup N)^*\langle K\rangle$ do not contain more than one 1 and they are all losing too. So, the set $x\langle(M \cup N)^*K\rangle$ contains only losing words, hence a contradiction.
2. W is monotone. Suppose by contradiction that W is not monotone. Then there exist $x, y \in [k]^*$ and $M, N \subseteq [k]^*$ such that $xM < xN$ and $yN < yM$. So, xM and yN contain only losing words, xN and yM contain a winning word. If x contains more than one 1, all words in the first two sets are losing, hence a contradiction. If x contains one 1, then no word in M contains 1. However, there is a winning word in yM, so y contains two 1's. Hence, yN contain only winning words, which is a contradiction. If x does not contain a 1, there is a word in N with two 1's. Hence, yN contains at least a winning word, which is again a contradiction.
3. $\overline{W}$ is selective. Suppose by contradiction that $\overline{W}$ is not selective. Then, there exist $x \in [k]^*$ and $M, N, K \subseteq [k]^*$ such that $x\langle(M \cup N)^*K\rangle = x\langle(M \cup N)^*\rangle \cup x(M \cup N)^*\langle K\rangle$

contains a winning word and $x\langle M^*\rangle \cup x\langle N^*\rangle \cup x\langle K\rangle$ contains only losing words. Observe that no word in M or N does not contain 1, else if $m \in M \cup N$ does not contain a 1, $xm^\omega \in x\langle M^*\rangle \cup x\langle N^*\rangle$ does not contain 1's and it is a winning word. So the words in the set $x\langle (M \cup N)^*\rangle$ contain infinitely many 1's and they are losing. Moreover, since $x\langle K\rangle$ contains more than one 1, the words in $x(M \cup N)^*\langle K\rangle$ contain more than one 1 and they are all losing. So, the set $x\langle (M \cup N)^* K\rangle$ contains only losing words, hence a contradiction.

4. $\overline{W}$ is monotone. Suppose by contradiction that $\overline{W}$ is not monotone. Then there exist $x, y \in [k]^*$ and $M, N \subseteq [k]^*$ such that $xM < xN$ and $yN < yM$. So, xM and yN contain only losing words, xN and yM contain a winning word. If x contains more than one 1, all words in the first two sets are winning, hence a contradiction. If x contains one 1, then there is a word in N that does not contain 1's. Since yN contains only losing words, y contains more than one 1. So, all words in yM are losing, hence a contradiction. If x does not contain 1, then all words in M contain more than one 1, so all words in yM are losing, hence a contradiction. □

5 Strong Selectivity

We proved in the previous section that determinacy, strong monotonicity, and strong concavity do not constitute a characterization for half-positionality. Indeed, we observe that strong concavity is a stronger property than what is needed. It asks that no matter how two losing words are interwoven the results is still a losing word. The aim is to relate the alternation of player 0 between two behaviors to the switching between the two words. However, on a game graph, player 0 is not actually free to switch at every point between the two behaviors, but only at particular nodes where the two paths meet. Moreover, since a word should represent a positional behavior for player 0 on a finite graph, it should contain some periodicity. Both these requirements are achieved through the property of selectivity. However, selectivity can only be used in the hypothesis player 1 is using a positional strategy, since only in that case player 0 makes use of regular behaviors. In order to incorporate into selectivity, the ability to evaluate also non regular sets of words, we define a new stronger version of it.

Definition 10 A goal W is *strongly selective* if and only if for all $x \in [k]^*$ and for all languages $M, N, K \subseteq [k]^*$ we have that $x\langle (M \cup N)^* K\rangle \leq x\langle M^*\rangle \cup x\langle N^*\rangle \cup x\langle K\rangle$.

Selectivity and strong selectivity represent two weaker properties than strong concavity.

Lemma 14 *All strongly concave goals are strongly selective.*

Proof For all words $x \in [k]^*$, for all languages $M, N, K \subseteq [k]^*$, we have that $x\langle (M \cup N)^* K\rangle \subseteq x((\langle M^*\rangle \otimes_s \langle N^*\rangle) \otimes_s \langle K\rangle) \leq x\langle M^*\rangle \cup x\langle N^*\rangle \cup x\langle K\rangle$. □

Unfortunately, the strong versions of selectivity and monotonicity prove not to be sufficient conditions to half positionality[2].

[2] We thank Hugo Gimbert and Wiesław Zielonka for pointing out the counterexample.

Lemma 15 *There exists a strongly monotone and strongly selective goal which is not half-positional on initialized games.*

Proof For all colors $i \in [k]$ and finite paths π, let $|\pi|_i$ be the number of edges colored by i on π, and let $|\pi|$ be the number of edges in π. Moreover for all $n \in \mathbb{N}$ let $\pi^{\leq n}$ be the prefix of length n of π. The strongly monotone and strongly selective goal is the set W of all the infinite words m such that, for all colors $i \in [k]$, the limit $\lim_{n\to+\infty} \frac{|m^{\leq n}|_i}{|m^{\leq n}|}$ exists and is finite. The goal is prefix independent. Indeed, let $\pi = x\pi'$ then for all $i \in [k]$ we have $\lim_{n\to+\infty} \frac{|\pi'^{\leq n}|_i}{|\pi'^{\leq n}|} = \lim_{n\to+\infty} \frac{|\pi^{\leq n+|x|}|_i - |x|_i}{|\pi^{\leq n+|x|}| - |x|} = \lim_{m\to+\infty} \frac{|\pi^{\leq m}|_i}{|\pi^{\leq m}|}$. The goal is also strongly selective. Indeed, suppose by contradiction that there exist a sequence $x \in [k]^*$, and three languages $M, N, K \subseteq [k]^*$ such that $x\langle(M\cup N)^*K\rangle$ contains one winning word and $x\langle M^*\rangle \cup x\langle N^*\rangle \cup x\langle K\rangle$ contains only losing words. In this case, M and N must be empty else any periodic word $\pi = m^\omega \in M^* \cup N^*$ with $m \in M \cup N$ has a finite limit $\lim_{n\to+\infty} \frac{|\pi^{\leq n}|_i}{|\pi^{\leq n}|} = \frac{|m|_i}{|m|}$, for all colors i. So, the set $\langle x(M\cup N)^*K\rangle = x\langle K\rangle$ and contains only losing words which is a contradiction. The above goal is not half-positional in the arena $(\{u\},\{v\},\{(v,0,u),(v,1,u),(u,0,v),(u,1,v)\})$ with $k = 1$ (Fig. 1(c)), starting from any node. Player 0 can win with a strategy with memory by choosing from v to u the opposite of the color that player 1 chose from u to v right before, thus yielding a path in $[k]^*(10)^\omega$ which has limit $\frac{1}{2}$ for both colors. However if player 0 uses a positional strategy, it will only choose one color from v to u, let suppose without loss of generality that he chooses color 0. The player 1 can force a path $\pi = \prod_{i=0}^{+\infty}(00)^{2^i}(10)^{2^i}$. Then we have $|\prod_{i=0}^{l}(00)^{2^i}(10)^{2^i}| = \sum_{i=0}^{l} 4\cdot 2^i = 4(2^{l+1}-1)$, and $|(\prod_{i=0}^{l-1}(00)^{2^i}(10)^{2^i})\cdot(00)^{2^l}| = 4(2^l + 2^{l-1} - 1)$. Moreover, $|\prod_{i=0}^{l}(00)^{2^i}(10)^{2^i}|_1 = \sum_{i=0}^{l} \cdot 2^i = (2^{l+1}-1)$, and $|(\prod_{i=0}^{l-1}(00)^{2^i}(10)^{2^i})\cdot(00)^{2^l}|_1 = \sum_{i=0}^{l-1} \cdot 2^i = 2^l - 1$. So we have $\frac{|\prod_{i=0}^{l}(00)^{2^i}(10)^{2^i}|_1}{|\prod_{i=0}^{l}(00)^{2^i}(10)^{2^i}|} = \frac{1}{4}$, and moreover $\frac{|(\prod_{i=0}^{l-1}(00)^{2^i}(10)^{2^i})\cdot(00)^{2^l}|_1}{|(\prod_{i=0}^{l-1}(00)^{2^i}(10)^{2^i})\cdot(00)^{2^l}|} = \frac{2^l-1}{3(2^l-1)+2(2^l)} = \frac{2^l-1}{5(2^l)-3} = \frac{2^l-\frac{3}{5}}{5(2^l)-3} - \frac{\frac{2}{5}}{5(2^l)-3} < \frac{1}{5}$. This shows that in the limit $\frac{|\pi^{\leq n}|_1}{|\pi^{\leq n}|}$ oscillates between $\frac{1}{4}$ and something less than $\frac{1}{5}$. □

Although the following theorem is obtained easily from the techniques developed in [4], we think that it is worth mentioning that half positionality on arenas controlled only by player 0 is equivalent to the selectivity of the goal. Since the selectivity is similar in a way to strong concavity, we show that strong concavity is a condition useful to assert that, on decisions independent from player 1, player 0 prefers a fixed behavior rather than switching between two different ones. We prove the above statement by making use of the following lemma proved in [4].

Lemma 16 **([4])** *Let A be a finite co-accessible* [3] *automaton recognizing a language $L \subset [k]^*$ and having starting state q. Then, $\langle L\rangle$ is the set of infinite color sequences on the graph of A starting in q.*

Theorem 4 *A goal is selective if and only if it is half-positional on all arenas controlled by player* 0*.*

[3] An automaton is co-accessible if and only if from every state there is a path reaching an accepting state. It's easy to see that a minimized automaton is co-accessible.

Proof [only if] Suppose that a goal W is half-positional on all game graph controlled by player 0 but non-selective. Let $x \in [k]^*$ and $M, N, K \subseteq [k]^*$ be three recognizable languages such that $x\langle (M \cup N)^* K \rangle \not\subseteq x\langle M^* \rangle \cup x\langle N^* \rangle \cup x\langle K \rangle$. This means that there is a winning word in $x\langle (M \cup N)^* K \rangle$ and $x\langle M^* \rangle \cup x\langle N^* \rangle \cup x\langle K \rangle$ contains only losing words. Let G_x, G_M, G_N be the minimized finite automata recognizing the languages $\{x\}, M, N$, respectively, and having only one starting state with no transition returning to it and one final state with no transition exiting from it. Let G_K be the minimized finite automaton recognizing the language K, having only one starting state with no transition returning to it. We construct the game graph G by combining together the graphs G_x, G_M, G_N, G_K. Precisely we glue together the final state of G_x, the initial and final states of G_M and G_N and the initial state of G_K in a new node t. Observe that, by gluing together the initial and final states, the automata G_M, G_N recognize M^* and N^*, respectively. The initial state of G is the starting state of G_x. Thus the graph G recognizes the language $x(M \cup N)^* K$. Hence by Lemma 16, every infinite path in G is in $\langle x(M \cup N)^* K \rangle = x\langle (M \cup N)^* K \rangle$. Since this set contains a winning word, there is a winning strategy for player 0. However, if player 0 uses a positional strategy he cannot win. Indeed, player 0 reaches first the node t by constructing the color sequence x on G_x. In the node t player 0 chooses once and for all which of the subgraphs G_M, G_N, G_K he will use, so the infinite play will be of the form xm where m is an infinite path in G_M,G_N or G_K. By Lemma 16, $xm \in x\langle M^* \rangle \cup x\langle N^* \rangle \cup x\langle K \rangle$. But this set contains only losing words. Hence, xm is losing.

[if] Suppose that a goal W is selective, we prove by induction on the number of edges exiting from the nodes of the arena G controlled by player 0 that if there exists a winning strategy for player 0 then there exists a positional one. As base case there exists only one edge exiting from the nodes of G, hence player 0 has only one strategy, which is trivially positional. Suppose that in the arena there are n edges exiting from nodes of player 0 and that for all graphs with at most $n-1$ edges exiting from nodes of player 0, if player 0 has a winning strategy he has a positional one. Let t be a node of player 0 in G such that there is more than one edge exiting from t. We can partition the set of edges exiting from t in two disjoint non-empty sets E_α and E_β. Let G_α and G_β be the two subgraphs obtained from G by removing the edges of E_β and E_α, respectively. There are two cases to discuss. First, suppose that either in G_α or G_β player 0 has a winning strategy. Then, by inductive hypothesis he has a positional winning strategy. It is easy to see that such a strategy is winning in G too, indeed player 0 is able to play always in G_α or G_β since he controls every node.

Suppose now that player 0 has no winning strategy in G_α and in G_β. We prove the thesis by showing that player 0 has no winning strategy in G. Let M_α and M_β be the sets of all finite color sequences from t to t and K_α and K_β be the sets of all finite color sequences starting from t, in G_α and G_β, respectively. Such sets are regular languages: M_α and M_β are recognized by the automata having respectively G_α and G_β as state graphs, with starting node t and accepting set $\{t\}$. The sets K_α and K_β are the languages accepted by the automata with state graphs G_α and G_β, respectively, with starting node t and accepting set given by all the states.

Suppose now by contradiction that there exists a winning strategy for player 0 in G. Then this strategy will form a winning path π. Such a path cannot be in G_α or G_β, or else player 0 has a winning strategy in one of those subgraphs. So the path is in G

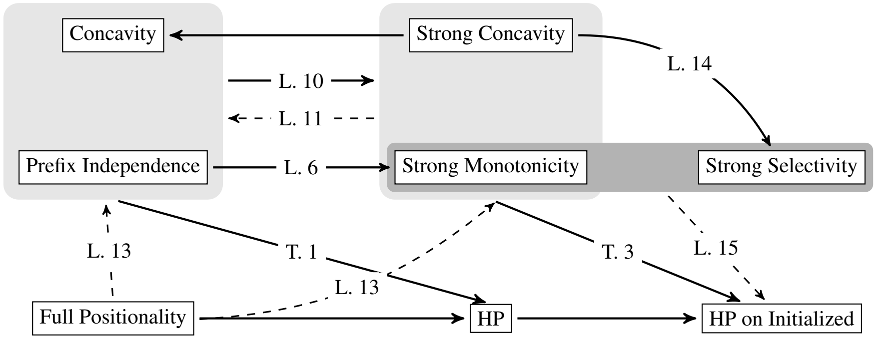

Fig. 2 Summary of results. Continuous arrows represent a holding implication and dashed ones a false one. Arrows are labeled with the corresponding lemma or theorem. Moreover, a gray box represents a conjunction of conditions.

and passes through t. Let x be the shortest prefix of π ending in t, then π belongs to the set $x\langle(M_\alpha \cup M_\beta)^*(K_\alpha \cup K_\beta)\rangle$, since it starts with x, then either loops forever from t to t in G_α and G_β, or possibly ends with an infinite path that never comes back to t. However, for $\gamma \in \{\alpha, \beta\}$, the sets $x\langle M_\gamma^*\rangle$ and $x\langle K_\gamma\rangle$ contain only paths in G_γ, so they are losing. Thus, we have $x\langle(M \cup N)^*K\rangle \not\lesssim x\langle M^*\rangle \cup x\langle N^*\rangle \cup x\langle K\rangle$, which contradicts selectivity. □

6 Conclusions

In this paper, we defined a new sufficient condition for half-positionality on finite arenas, which turns out to be strictly weaker (i.e., broader) than that defined by Kopczyński in [6], as long as determined goals are considered. We discussed the conditions presented in [4] for full-positionality and we proved that a stronger partial form of them does not ensure half positionality. Figure 5 contains a graphical representation of our main results, both positive and negative.

The main open problem left by this research is the formulation of a complete characterization of half-positionality. Our attempts lead us to believe that this may be no easy task. Another interesting question for further research is whether or not the properties of strong monotonicity and strong concavity imply determinacy. The answer to this question may simplify the statement of Theorem 3 by removing the hypothesis of determinacy.

Finally, another open problem consists in developing algorithms for checking whether a goal, given in input in some effective way such as an automaton or a temporal logic formula, satisfies the conditions outlined in this paper and is therefore HP. Such an algorithm may be used as a preliminary step in controller synthesis tools, in order to estimate the amount of memory that the synthesized controller will need. In other words, a controller synthesis tool could take as input a control goal expressed by a temporal logic such as LTL [9], check whether the given goal is half-positional, and warn the user (or even refuse to synthesize the controller) if it is not. For prefix-independent goals, Kopczyński proved that checking the half-positionality of a goal expressed by a parity automaton is computable in exponential time (w.r.t. the size of the automaton) [7].

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