University of Naples
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Faculty of *Mathematical, Physical and Natural Sciences*

Department of *Physical Sciences*

Master thesis in theoretical physics:

"Discrete quantum field theories of the gravitational field"

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"Mathematics is the tool specially suited
for dealing with abstract concepts of any kind,
and there is no limit to its power in this field.
For this reason, a book on the new physics,
if not purely descriptive of experimental work,
must be essentially mathematical."

P.A.M. Dirac

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Introduction

Three out of the four interactions provided in nature, namely the electromagnetic, the weak and the strong interactions, are well described by the so-called Standard Model, which is a Yang–Mills gauge theory with gauge group $G = \mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$. The fields are regarded as representations on which the gauge group acts, and the requirement for the Yang–Mills Lagrangian to be gauge invariant translates into the existence of pure gauge fields, propagating and coupling both to the matter fields and to themselves. The propagation is achieved through the exchange of gauge bosons (regarded as force carriers of the interactions) in number equal to the number of generators of the gauge group. Such a description does not provide the masses of the elementary particles inside the model, so a procedure giving raise to well defined values for the masses must be introduced: this is obtained by means of the introduction of an external field coupling to each term in the Yang–Mills Lagrangian, the Higgs field. This further field is defined in a way such that it has degenerate ground states, connected one another by means of the action of the gauge group. The choice of a particular vacuum state and the rescaling of all the fields with respect to that ground state introduces the phenomenon of spontaneous symmetry breaking; the gauge and the matter fields couple quadratically to the Higgs field, so massive terms arise. In this way each field is provided with its own mass; recall that the search for the mass of the Higgs field itself is still an open question.
The coupling constant of each interaction generally depends on several quantities. This can be seen by studying the equations of the renormalization group, in particular the running coupling constant flow $\beta(g)$. This equation shows that the behaviour of the coupling constant depends on the energy scale of the interaction, the mass of the gauge bosons, and, in general, on the specific way the interaction couples to matter and to the gauge fields. Thus forces in nature are not inherently strong or weak, but they *appear* more or less strong accordingly to the energy scales one is looking at. As a consequence, if an observer works at fixed (low) energies, the electromagnetic, the weak and the strong interactions appear as distinct forces, because the coupling constants are very far to be close and related one other. On the contrary, if we take the high energy limit for the coupling constants representing these three fundamental forces, we obtain, from the equations of the renormalization group, that they approach the same value; this is a very

important feature which leads us to consider the electromagnetic, the weak and the strong interaction as different aspects of a general and unified theory.

The other interaction provided in nature, the gravitational force, is quite different from the other ones. The principle of equivalence states that the coupling constant appearing in the expression of the potential energy for a particle in a gravitational field is the same appearing in the kinetic term in its free Lagrangian; morally speaking, the charge generating the gravitational force, namely the gravitational mass, is the same inertial mass appearing in the Newton law. This feature characterizes the dynamic of a particle submitted to a gravitational field as a dynamic depending only on the metric properties of the space-time. This principle naturally leads to General Relativity, in the form of the Einstein field equation for the metric tensor g on a manifold M representing space-time.
In this way, while the Standard Model describes the electromagnetic, the weak and the strong interactions as fields *on* the space-time, General Relativity describes gravitation as a field theory *of* the space-time. The configuration space emerges as the space $\mathcal{T}_2^0$ of all (metric) tensor fields on a Riemannian manifold M.

The picture arising in physics is then made, on one hand, by the quantum field theoretic approach of the Standard Model, and, on the other hand, by the framework of General Relativity. The Standard Model of electromagnetic, weak and strong interactions is characterized by an extraordinary empirical success, nearly unique in the history of science. For this reason, apart from the question of the Higgs mechanism or something related to it, we are attempted to believe that the Standard Model is the very close to be the correct answer to the question of the description of these three interactions.
On the other hand, General Relativity emerges as a classical field theory on a fibre bundle, so it is not *quantum*. Many experimental results and predictions have been accomplished by means of the framework of General Relativity, as, for example, the deflection of light rays in a gravitational field, the redshift of frequences emitted by pulsars or the anomalous perihelion shift of the orbit of the planet Mercury. Nevertheless, there exists some reasons leading us to suppose that General Relativity, in the original way formulated by Einstein, could not give the correct understanding of the gravitational interaction. For

example, some classes of solutions of the Einstein equations present singularities on very small scales, showing that something in this theory goes wrong when we try to describe space-time at certain lenghts; there exist also some cosmological issues left unresolved within the framework of classical General Relativity, as the existence of an entropy associated with black holes, or, in general, the full statistical mechanics provided by the black holes thermodynamics.
Apart from these arguments, the key point in this picture is the coupling between quantum gauge and matter fields, provided by the Standard Model, with the gravitational field: General Relativity has changed the notions of space and time too radically to agree with quantum mechanics, which is a theory formulated *on* the space-time. This will not be achieved unless gravitation will be incorporated into a unified theory including the other three interactions. A *possible* answer to this question is the attempt to formulate a *quantum theory* of gravitation, because, from the purely conceptual point of view, we expect also gravitation to be formulated as a quantum theory, whose classical limit is General Relativity. At this point is very important to remark that, unless a complete theory of the gravitational field agreeing with the Standard Model will be found, there is no reason to regard the quantum gravity as *the right* way to achieve this unification. Quantum gravity is only a *possible* response to this question.
Nevertheless it is important to stress, at this point, that physicists have also provided another tool in order to describe in an unified mechanism all the interactions existing in nature: string theory. This also appears as a candidate for a complete description of matter and gauge fields, since it incorporates gravity in its formulation.

Let us now focus the attention on the problem of quantum gravity. Any general quantum theory of gravitation is expected to work at very small scales, indeed the Planck scales provided by the fundamental constants $G, \hbar$ and c. If we arrange this constants in order to give raise to a length scale, we find that the correct arrangement is

$$L_P^{(grav)} = \left(\frac{G\hbar}{c^3}\right)^{1/2} \sim 10^{-35}\ m$$

so it really works at very small scales (recall that the characteristic "length" for quarks is $\sim 10^{-18} m$)! The presence of singularities in the solutions of the

Einstein field equations may suggest the possibility for the space-time to be not continuum at some scales, for example, at scales of order of $L_P^{(grav)}$. This corresponds to times of the order of

$$T_p^{(grav)} = \frac{L_P}{c} \sim 10^{-44}\ s$$

which is the time a photon would take to cross a distance $L_P^{(grav)}$. This is indeed the regime expected for quantum field theory and gravity to *work together*, in the sense that they would be equally strong. For this reasons, at least if one considers physical systems at such scales (for example the Big Bang and other singular solutions of the Einstein equations), a quantum treatment for the gravitational field is required. Thus, any generical quantum theory of gravitation must be consistent, at some regimes, with quantum mechanics and General Relativity, emerging as limits, at some scales, of this more general theory. Consistency with quantum mechanics and General Relativity is indeed an extremely strict constraint for quantum gravity.

There exist several approaches to the quantum theory of gravitation, and we do not claim to furnish here a complete and full list of arguments leading to each of these different formulations; the reader interested in such area can find a history of the various approaches to quantum gravity in [19]. On the contrary, our purpose is to introduce and describe one particular method of quantization of the gravitational field, namely the discrete quantum field theory approach.
The canonical quantization procedure for a field theory fails to work with the gravitational field if we want to quantize the metric tensor field $g_{\mu\nu}$, because it is not renormalizable at the perturbative level. This is because the coupling constant G has not the suitable dimensions required for a field theory to be renormalizable. As a consequence, the number of divergences in the Feynman amplitudes will increase according to the order of the perturbation expansion one looks at, so the perturbative series for the generating functional does not converge. Thus, the path integral quantization à la Feynman cannot furnish the correct answer. One is then led to try to implement the quantization using the Dirac old–quantization procedure, i.e. "translating" the observables on the phase space into self-adjoint linear operators acting on some Hilbert space $\mathcal{H}_{kyn}$ of square integrable functions; the "quantum states" of the theory emerge as the subspace $\mathcal{H}_{dyn} \subseteq \mathcal{H}_{kyn}$ which satisfy the

constraint imposed on the theory. The search for such dynamical states will turn out to be a hard work, because of the presence of difficult technical details required in solving the constraints equations; a more detailed analysis of this problem is in [6].
In this paper we try to introduce a different procedure leading to the quantum theory of gravitation, with the idea to solve the problem of the non-renormalizability by introducing a different procedure of regularization: a *discrete* quantum field theory based on General Relativity. The underlying idea is quite simple: if we have a flat space-time, take it two-dimensional for simplicity, we can introduce a triangulation in each local neighbourhood, the triangles all meeting in a common vertex. Then, if one covers a closed circle surrounding the vertex, it encloses an angle of 2π; if the space-time is locally non-flat in the vertex, the same procedure encloses an angle greater or smaller than 2π, according to whether the manifold is locally convex or concave. In this way one can take into account the local curvature of the manifold by assigning a defect angle, (formally an element of the group U(1)), to each closed curve surrounding a vertex (i.e. to each "loop"), referring to the failure of the closed curve to sum up to 2π.

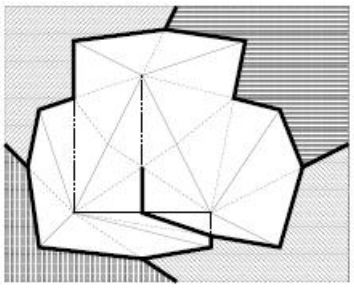

Figure 1: *A triangulation on the space-time.*

Moreover, by turning the Feynman integrals into discrete sums over these "latticed" manifolds, one can avoid the divergences appearing in the perturbative expansion, without inserting cutoffs into the physical quantities. In this way, the regularization is obtained by acting on the space-time itself, instead of acting on the fields. We want once more to remark the word *discrete* in this context: it means that the carrier manifold M, representing space-time, is regarded as a set of purely combinatorical objects, made up of graphs labelled with different representations of the gauge group G. These algebraic objects are the so-called *spin networks*, and furnish the quantum

states of the theory. Evolution between spin networks is provided by a *spin foam*, a 2-complex κ bounded by two spin networks. The collection of spin networks and spin foams of a defined kind emerges as a particular geometry of the space-time. These spin foams are regarded as a sort of "Feynman diagrams" emerging from a well defined action for the field, written in terms of algebraic quantities related to the gauge group G; the sum over all possible Feynman diagrams reproduces the generating functional Z_M for the field. The search for a gauge invariant action, which can reproduce spin foams as Green's function, is the aim of the Group Field Theory.
Apart from these considerations, we ought to say that the final answer to each quantum theory is given by the study of the renormalization group. In fact the equations of the flows furnishes the physical behaviour of the observable quantities one is expected to detect, namely the running coupling constant, the corrections to the bare propagators and to the vertex amplitudes and the asymptotic freedom. We have not dealt with these problems in this work.

In the specific this thesis is organized as follows: the first chapter provides a description of the classical theory of the gravitational field as a gauge theory on a principal bundle $P(M, G)$. The Einstein–Hilbert action is written in terms of the tetrad fields on the space-time, and subsequently it is shown that this description can be achieved by introducing a more general action, the BF-action, imposing constraints at the level of the fields involved, the constraint taking into account the physical degrees of freedom of the field. Then, the invariance under transformations provided by the gauge group suggests us to write down a full invariant action, the Palatini–Holst action, differing from the Einstein–Hilbert one for the introduction of the so-called Immirzi parameter. The Ashtekar variables are then introduced to write the Palatini–Holst equations in a more compact form.
The second chapter is devoted to the introduction of the quantum states in the spirit of a canonical quantization procedure. We describe the full theory of spin networks, defined out of a discrete regularization of the carrier manifold M, and we show how to calculate the evolution between quantum states (i.e. transition amplitudes) by means of spin foams, depending on the chosen topology of the space-time. The sum over all possible spin foams turns out to be the transition amplitude between two different *geometries* of the space-time, each geometry being regarded as a state of the gravitational field.

In the third chapter we regard the spin networks as purely combinatorical objects, not necessarily related to the structure of the space-time, and we calculate the generating functional for generic D-dimensional BF-theories on such algebraic objects. Moreover, in the most interesting case of 4 dimensions, two different models are presented, namely the Barrett-Crane model and the EPRL/FK model, each one of them trying to implement the constraint imposed on the BF-theory in order to reduce to General Relativity, in a different way.
The fourth chapter contains an introduction to the Group Field Theory. This is indeed the key result of this work: we introduce a quantum field theory whose fields act on a group manifold. The transition amplitudes are calculated using the standard tools and techniques of quantum field theory, and the results we obtain are exactly the same found using the discrete approach of spin networks.
Finally, an appendix is provided for some mathematical tools, in order to give a better understanding of the subject presented in the paper. In particular, we introduce the theory of connections on principal bundles, giving the proofs of some theorems and properties used in the thesis.

Chapter 1

Classical gauge theory of the gravitational field

"It is as we had observed in the ocean many animals living on an island: animals on an island. Then we discover that the island itself is in fact a great whale. Not anymore animals on the island, just animals on animals. Similarly, the universe is not made by fields on spacetime; it is made by fields on fields."

C. Rovelli

In this chapter the gravitational field is described as a gauge theory on a principal bundle $P(M, G)$, whose carrier manifold M is identified with the space-time, and whose total space P with the tangent bundle TM over M. The Einstein–Hilbert action is first written in terms of the tetrad fields and subsequently it is shown that a particular action (BF-type) can give raise to the same equations of motion if one add a constraint term in the Lagrangian. At this point such a description is achieved by using a new set of variables, namely the holonomies and the fluxes, constructed out of the so-called "Ashtekar variables"([6]), which can take into account the topological structure of the manifold M used as a carrier space for the field theory.
This description of the gravitational field in terms of holonomies will be the starting point in order to quantize the theory.

1.1 Construction of a metric theory

Let M be an n–dimensional differentiable manifold (which we identify with the space-time) and let T_mM be the tangent space to M in $m \in M$; taken a chart (U_i, ϕ_i) such that $m \in U_i$, a basis for the tangent space T_mM is given by the set $\left\{\underline{e}_\rho(m)\right\}$, $\rho = 0, \ldots, n-1$.
Take N to be a vector field on M and let $m \equiv m(s)$ be the flow induced by such a field, namely $m(s)$ is the subset of points of M whose representation in the chart (U, ϕ) satisfies:

$$\begin{cases} \dfrac{d}{ds}\, \phi\left(m(s)\right)^{\,\mu} &= N^\mu\left(\, m(s)\,\right) \\ \\ \phi\left(m(0)\right)^{\,\mu} &= \phi_0^\mu \end{cases} \tag{1.1.1}$$

$N^\mu\left(\, m(s)\,\right)$ being the components of the vector field N with respect to the basis $\left\{\underline{e}_\rho\right\}$ when it is evaluated in $m(s)$ using the chart (U, ϕ). The solution of (1.1.1) can be represented (using the above chart) in the form

$$\phi\left(m(s)\right)^{\,\mu} = e^{\,sN}\ \phi_0^\mu. \tag{1.1.2}$$

The flow induced by N is a one–parameter subset of M that we define to be $I \subset M$.
Now take $n-1$ vector fields on M, namely $(S_1, \ldots, S_{n-1})$ and let $\Sigma \subset M$ be the set of points $m \in M$ such that a basis for the tangent space $T_m\Sigma$ is given by $\{S_a(m)\}$, $a = 1, \ldots, n-1$. Furthermore we require $N(m) \notin T_m\Sigma$.

We want the carrier manifold M to be expressed as

$$M = I \times \Sigma \tag{1.1.3}$$

in order to be a sort of "space-time". We call (1.1.3) an "observer", because it allows a decomposition of the space-time in space *and* time, which is just what we require an observer to do. In the following we try to describe the gravitational field in 4 dimensions; in that case it must be $dim\ \Sigma = 3$, and, as a consequence, $dim\ M = 4$. Nevertheless, theories in lower dimensions also exist. Equation (1.1.3) states that each point $m \in M$ is of the form:

$$m = (m_1 \in I; m_2 \in \Sigma) \tag{1.1.4}$$

but its representation in the charts (U_i, ϕ_i) and (U_j, ϕ_j), with $m \in U_i \cap U_j$, must change according to a well defined group of transformations G (identified with the Poincaré group); namely we must require

$$\phi_j(m) = G^{(i)}_{(j)}\ \phi_i(m) \tag{1.1.5}$$

so the set of transition functions $\psi^{(i)}_{(j)}$ on the manifold M must be nothing that the Poincaré group. Notice that the principles of general relativity state that each physical theory must be described in the same form by using both the chart U_i and the chart U_j, so each equation must not depend on which representation of the point m in (1.1.5) is chosen.

The splitting of M as in (1.1.3) allows to write the tangent space T_mM as a direct sum

$$T_mM = T_{m_1}I \oplus T_{m_2}\Sigma \tag{1.1.6}$$

and then each vector in m could be written as

$$X(m) = n\ N(m) + t\ T(m) \tag{1.1.7}$$

with $N(x) \in T_{m_1}I;\ T(x) \in T_{m_2}\Sigma$. Using the basis of $T_{m_2}\Sigma$ the above equation could be put in the form

$$X(m) = n\ N(m) + U^a\ S_a(m). \tag{1.1.8}$$

It is common use to refer to the coefficients n and U^a in the above equation as to the lapse and shift coefficients of the vector $X(m)$. A splitting of the tangent space T_mM, at each point $m \in M$, can be formally achieved by introducing a Lie-algebra valued one-form $\omega \in \mathfrak{g} \otimes T^*M$, called the connection one-form, such that one of the vector spaces realizing condition (1.1.6) could be regarded as the kernel of ω. For example we can define the tangent (horizontal) subspace $T_{m_2}\Sigma$ as the set of vectors $T(m)$ satisfying the condition $\omega\,(\ T(m)\ ) = 0$. On the contrary, the subset $T_{m_1}I$ could be chosen in order to satisfy $\omega\,(\ T_{m_1}I\ ) = \mathfrak{g}$. It is always possible ([14]) to choose ω in order to have a vector space isomorphism between $T_{m_1}I$ and $\mathfrak{g}$. We choose the gauge group to be $G \equiv \mathrm{SL}(2, \mathbb{C})$ (we have in mind the Lorentz group $\mathrm{SO}(3, 1)$ but we want to introduce the theory starting from its covering $\mathrm{SL}(2, \mathbb{C})$) and refer to the one-form connection ω as to the spin-connection.

Because we want to construct a metric theory on the space-time, M must be endowed with a Riemannian structure. In particular we choose the vector fields N and S_a such that

$$\begin{cases} g_m\ (N(m)\ ,\ S_a(m)) & = 0 \\ g_m\ (N(m)\ ,\ N(m)) & > 0 \end{cases} \tag{1.1.9}$$

g being the Riemannian metric on M. With this choice we regard the fields N and S_a as orthogonal according to the metric g, and N to be "timelike" because the observer (1.1.3) wants to regard the flow I as the world-line of the dynamics. Without loss of generality we put $g_m\,(N(m)\ ,\ N(m)) = 1$. At any rate, the vector space T_mM also admits a representation in terms of the "covariant" basis $\left\{\underline{e}_{\ \mu}(m)\right\}$; in this basis the vector fields N and T decompose as $N(m) = N^\mu(m)\ \underline{e}_{\ \mu}(m)$ and $T(m) = T^\mu(m)\ \underline{e}_{\ \mu}(m)$ and the scalar products (1.1.9) become:

$$\begin{cases} g_{\mu\nu} N^\mu T^\nu & = 0 \\ g_{\mu\nu} N^\mu N^\nu & = 1 \end{cases} \tag{1.1.10}$$

With the help of these definitions the scalar product of any two vectors $\in T_mM$ can be written as

$$\begin{aligned} g_m\ (X_1(m)\ ,\ X_2(m)) &= g_m\left(n_1\ N(m) + U^a\ S_a(m), n_2\ N(m) + V^b\ S_b(m)\right) \\ g_m\ (X_1(m)\ ,\ X_2(m)) &= n_1\ n_2 + U^a\ V^b\ g_{\mu\nu}\ S_a^\mu\ S_b^\nu\ . \end{aligned} \tag{1.1.11}$$

We define the following functions of the metric:

$$q_{ab}\,(g) = g_{\mu\nu}\ S_a^\mu\ S_b^\nu \tag{1.1.12}$$

which will be useful later in the text. The condition for the vector field N to be time-like (equations (1.1.10)) thus allows us to write the scalar product between two vectors only by using these new functions of the metric. So, in the following, we refer in the same way to the components of the metric tensor $g_{\mu\nu}$ or to the functions q_{ab}, provided the (1.1.11) to be invertible. In fact, all the quantities related to the metric tensor appearing in the text will admit a more compact form in terms of the new variables q_{ab}.

The principles of general relativity allow us to construct the theory of the gravitational field as a metric theory (i.e. a theory whose dynamical field

is the metric tensor on a differentiable manifold), so we try to construct an action principle for the field led by ideas of general consistency. The action must be of the form

$$S\left[\ g\ \right] = \int_{space-time} (4-form)\,\text{on the space-time}\ \ . \tag{1.1.13}$$

We require the action to be invariant under the transformations induced by a group G, in particular we identify G as the Poincaré group. Furthermore S must be invariant under a reparametrization of the representation $\phi(m)$ and must take into account the effect of the geometry, so we should choose the 4–form such that it is derived from the Riemannian tensor on M, which is the mathematical object taking care of the effect of the curvature. In order to define the Riemannian tensor on the space-time, we need a notion of covariant derivative of vector fields. This is easily provided by the structure of a principal bundle $P\left(M,G\right)$, in particular the way this definition is induced by means of the connection one-form ω is shown in § A.5. Once one has achieved this result, we can put

$$S\left[\ g\ \right] = \alpha \int_M d^4x\sqrt{|g|}\ f(\mathcal{R}) \tag{1.1.14}$$

where $d^4x\sqrt{|g|}$ is the invariant volume element on M and $\mathcal{R}$ is the curvature scalar $\mathcal{R} = g\left(Ricci\right) = g^{\mu\nu}\ Ric_{\mu\nu}$. The simplest choice is $f\left(\mathcal{R}\right) = \mathcal{R}$, so we get

$$S\left[\ g\ \right] = \alpha \int_M d^4x\sqrt{|g|}\ \mathcal{R}\ \ . \tag{1.1.15}$$

We call (1.1.15) the Einstein-Hilbert action, because it gives raise, by making variations, to the Einstein equation of motion for the gravitational field, provided $\alpha = 1/(16\pi G)$.

Using the definition $q_{ab}\left(g\right) = g_{\mu\nu}\ S_a^\mu\ S_b^\nu$ one could rewrite the action $S\left[\ g\ \right]$ in terms of the new functions of the metric $q_{ab}\left(g\right)$. Notice that these new functions *cannot* be regarded as a sort of induced metric on a submanifold of M, even if the indices a, b take only the "spatial" values $1, 2, 3$. As a consequence of the invariance under reparametrization, the energy density associated with such an action vanishes identically "on-shell", i.e. if evaluated on the dynamical trajectories of the field. In particular, using the definition of $E_{\mathscr{L}}$

$$E_{\mathscr{L}} \stackrel{def}{=} \partial_0\left(g_{\mu\nu}\right)\frac{\partial\mathscr{L}}{\partial\left(\partial_0 g_{\mu\nu}\right)} - \mathscr{L} \tag{1.1.16}$$

one can rewrite the Lagrangian $\mathscr{L} = \sqrt{|g|}\ \mathcal{R}$ by using the functions $q_{ab}(g)$ and evaluate the energy. Using the definition of extrinsic curvature as

$$K_{ab} \stackrel{def}{=} \frac{1}{2n}\left(\partial_0 q_{ab} - L_U\ q_{ab}\right) \tag{1.1.17}$$

where n is the parameter which appears in the scalar product (1.1.11) as the coefficient of the "timelike" vector field N and L_U is the Lie derivative with respect to the "spatial" vector field appearing in (1.1.8), one finds (see [6])

$$\frac{\partial \mathscr{L}}{\partial\left(\partial_0 q_{ab}\right)} \stackrel{def}{=} \pi^{ab}(g, \partial_0 g) = \sqrt{|q|}\left(K^{ab} - tr(K)\ q^{ab}\right) \tag{1.1.18}$$

so that the Lagrangian density can be written as

$$\mathscr{L} = \pi^{ab}\ \dot{q}_{ab} - U^a\ H_a - n\ H \tag{1.1.19}$$

and, as a consequence, the energy density results

$$E_{\mathscr{L}} = U^a\ H_a + n\ H \tag{1.1.20}$$

with

$$\begin{aligned} H_a &= 2\sqrt{|q|}\ \nabla_{\underline{e}_b}\left(\frac{\pi^b_a}{\sqrt{|q|}}\right) \\ H &= \frac{1}{\sqrt{|q|}}\left(\ q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd}\ \right) - \sqrt{|q|}\ \mathcal{R}\ . \end{aligned}$$

Notice that the covariant derivative enters these definitions because of the presence of the Lie derivative in π^{ab}. On the dynamical solutions for the field, equation (1.1.20) identically vanishes.
Remark that, by setting (1.1.10), we have reduced the dynamical degrees of freedom, from $(4\cdot 5)/2 = 10$ of $g_{\mu\nu}$ to $(3\cdot 4)/2 = 6$ of q_{ab}. The "lost" degrees of freedom emerge as constraints imposed on the norm of the vector fields N and T with equations (1.1.10). These constraints can be taken into account by treating the components (n, U^a) of the vector field (1.1.8) as indipendent fields on which the Lagrangian depends; so, writing the action in terms of (1.1.19) as

$$S\left[\ q\,(g)\ ,\ n\ ,\ U^a\ \right] = \alpha \int_M d^4x\ \left[\pi^{ab}\ \dot{q}_{ab} - U^a\ H_a - n\ H\right] \tag{1.1.21}$$

we can make variations also with respect to n and U^a. These quantities enter the action without any derivative, so act as Lagrange multipliers, and the equations of motions one derives by varying n and U^a are:

$$H_a = 0 \ \text{ and } \ H = 0 \ . \tag{1.1.22}$$

This is a simpler way to show that the energy (1.1.20) identically vanishes.

1.1.1 Symplectic structure

If one inverts equation (1.1.18) and expresses the derivatives $\partial_0 g_{\mu\nu}$ in terms of the conjugate momenta π^{ab}, this set of variables can be treated as indipendent fields, so a sort of symplectic structure arises, with canonical Poisson brackets between $q's$ and $\pi's$ of the form:

$$\left\{ q_{ab}\,(m) \ , \ \pi^{cd}\,(m') \right\} \ = \ \delta^a_c \ \delta^b_d \ \ \delta\,(\ \phi(m) - \phi(m')\ ) \tag{1.1.23}$$

$$\left\{ q_{ab}\,(m) \ , \ q^{cd}\,(m') \right\} \ = \ \left\{ \pi_{ab}\,(m) \ , \ \pi^{cd}\,(m') \right\} = 0 \ . \tag{1.1.24}$$

Because (1.1.20) is the energy density, the total energy written in terms of $q's$ and $\pi's$ (now we can call it the *Hamiltonian*) is

$$\mathscr{H} = \int_{\Sigma \subset M} d^3x \ \ (U^a \ H_a + n \ H) \ . \tag{1.1.25}$$

This can be split into two parts, namely a "space-like" part and a "time-like" part, respectively:

$$\mathscr{H} \ = \ \mathscr{H}_{space} + \mathscr{H}_{time} \tag{1.1.26}$$

$$\mathscr{H} \ = \ \int_{\Sigma \subset M} d^3x \ \ (\ U^a \ H_a \ ) + \int_{\Sigma \subset M} d^3x \ \ (\ n \ H \ ) \ . \tag{1.1.27}$$

These two contributions satisfy the following Poisson brackets with respect to the canonical variables (q, π) (see [6]):

$$\{\mathscr{H}_{space} \ , \ q_{ab}\} = L_U \ q_{ab} \qquad \left\{\mathscr{H}_{space} \ , \ \pi^{cd}\right\} = L_U \ \pi^{cd} \tag{1.1.28}$$

$$\{\mathscr{H}_{time} \ , \ q_{ab}\} = L_N \ q_{ab} \qquad \left\{\mathscr{H}_{time} \ , \ \pi^{cd}\right\} = L_N \ \pi^{cd} + \frac{1}{2} \ q^{cd} n H + $$
$$-2n\sqrt{|q|}\left(q^{ac}q^{db} - q^{ad}q^{cb}\right) \ \mathcal{R}_{ab} \tag{1.1.29}$$

The first equation states that the "space-like" part $\mathscr{H}_{space}$ is the generator of space diffeomorphisms on Σ, through the flow expressed by the Lie derivatives

with respect to the space-like field U. The second equation shows that the "time-like" part $\mathscr{H}_{time}$ induces in the same way time diffeomorphisms[1] (Lie derivative with respect to the time-like field N). We are then led to regard the constraints (1.1.22) as the generators of the space-time diffeomorphisms group *Diff* (M) on the dynamical solutions of the equations.

1.2 Tetrad formalism

A consequence of the principle of equivalence is that every reference frame can always be locally reduced to an inertial frame by means of a suitable coordinate transformation. This means that we can always perform a coordinate transformation on the metric tensor g on M in order that it appears as a Minkowskian metric tensor η at least in a chart (U, ϕ) on M. We can take into account this peculiarity of the gravitational field by introducing two $(1,1)$-type tensors on M, namely e_1 and e_2, such that g can locally be expressed as

$$g\,(m) = (\ e_1\ \otimes\ \ e_2\ )\,(m)\ \ (\eta) \tag{1.2.1}$$

with $e_1, e_2\ \in \mathcal{T}_1^1(M)$. The (1.2.1) writes (the capital letters are used to label indices that are raised and lowered by using a Minkowskian metric)

$$g_{\mu\nu}(m)\ dx^\mu \otimes dx^\nu = \left(e^I_\mu(m) dx^\mu \otimes \underline{e}_I\ \otimes\ e^J_\nu(m) dx^\nu \otimes \underline{e}_J\right)\ \left(\eta_{AB}(m)\ dx^A \otimes\ dx^B\right)$$

which becomes in components

$$g_{\mu\nu}(m) = e^A_\mu(m) e^B_\nu(m)\ \eta_{AB}(m)\ . \tag{1.2.2}$$

The coefficients $e^A_\mu(m)$ appearing in the equation above are the *tetrad* components of the *tetrad* tensor e. Equation (1.2.2) takes into account the principle of equivalence stating that the metric tensor g is completely defined if we know the tetrad coefficients $e^A_\mu(m)$, which represent a sort of shift from the flatness due to gravitational effects. Notice that it follows from equation (1.2.1) that $\det(g) = -(\ \det(e)\ )^2$.

At this stage we introduce the notion of covariant exterior derivative and curvature of a Lie-algebra valued r-form; these definitions will be shown to

[1] Notice that the second term in the right side of eq. (1.1.29) vanishes on the dynamical solutions of the equations of motion $H = 0$ and $\mathcal{R}_{\mu\nu} = 0$.

be useful in order to write an action principle depending on some topological quantities which, in some suitable cases, reduces to the Einstein–Hilbert action for the gravitational field.

Given a connection one-form ω on a principal bundle $P(M, G)$, we define the covariant exterior derivative, acting on Lie-algebra valued r-forms, as the operator D such that its action on $\varphi \in \Omega^r(M) \otimes \mathfrak{g}$ ($\Omega^r(M)$ being the set of all r-forms on M) is achieved by

$$D\ \varphi\,(X_1, \ldots, X_{r+1}) = d\ \varphi\left(X_1^H, \ldots, X_{r+1}^H\right)\ . \tag{1.2.3}$$

Notice that this definition strictly depends on the connection one-form through the choice of the horizontal proiection of a vector field appearing in (1.2.3). The covariant derivative F of the connection one-form ω is called the *curvature* of the connection, so we have $F = D\ \omega$; clearly if ω is a Lie-algebra valued one-form, F is a Lie-algebra valued 2-form on M.

Now we want to show that the equations of motion for the gravitational field can be derived using an appropriate action constructed out of the tetrad tensors e on M. In this case $dim\ M = 4$, so we need a 4-form to be integrated on the space-time; this 4-form should contain informations about the metric field (expressed by means of the tetrads) and the curvature of the connection $F = D\ \omega$, because we want a relation between the metric and the curvature of the bundle. We choose this 4-form as the wedge product $(e \wedge e) \wedge F$, so the action can be written as

$$S\,[\ \omega\ ,\ e\ ] = \int_M tr\,[\ (e\ \wedge\ e)\ \wedge\ F\ ]\ . \tag{1.2.4}$$

This form of the action is referred to be the Palatini–Einstein action. The equations of motion one derives by making variations read

$$e \wedge F(\omega) \ = \ 0 \tag{1.2.5}$$

$$(D\ e)\,(e, \omega) \ = \ 0\ . \tag{1.2.6}$$

The Gauss law (1.2.6) ensures the compatibility between the tetrads and the connection one-form. It may happen that equation (1.2.6) cannot be solved explicitly (in particular this is the case when the tetrad coefficients $e_\mu^A(m)$ are not invertible); as a consequence we are not able to find a relation

between ω and e in the form $\omega \equiv \omega(e)$. On the contrary let us suppose that $\omega^* \equiv \omega^*(e)$ is an explicit solution of (1.2.6). If we substitute this expression into the action (1.2.4) we find (see [6] and [3]) that it just reduces to the Einstein-Hilbert action

$$S[\,\omega^*\,,\,e\,] = \int_M d^4x\,(\,\det(e)\,)\;\mathcal{R} \;=\; \int_M d^4x\sqrt{|g|}\;\mathcal{R} \;\equiv\; S_{EH}\;. \quad (1.2.7)$$

In fact, if we subsitute ω^* in the expression for $F(\omega)$, we find that the curvature of the bundle can be expressed in terms of the Riemann tensor on M, $F(\,\omega^*(e)\,) \;\propto\; Riemann\;(e)$. In this way, the real dynamical equation is $e \wedge F(\omega) = 0$, because it takes into account the dynamical degrees of freedom of the free gravitational field, expressed as a relation between the tetrads (i.e. the metric) and the Riemann tensor on the space-time. In conclusion we can state that if the tetrads are *non-degenerate* then the Einstein-Hilbert action can be derived from the Palatini–Einstein action, and the degrees of freedom carried by the gravitational field are contained into the equation $e \wedge F(\omega) = 0$.

We could add to the action (1.2.4) every term having the same invariance properties under the action of the gauge group and which does not modify the equations of motion (so this term must be also invariant under the action of the group of diffeomorphisms *Diff* (M)). The gauge group used to construct the field theory must be $G = \mathrm{SL}(2,\mathbb{C})$, so the trace operator appearing in (1.2.4) must be constructed out of the Lie-algebra $\mathfrak{sl}(2,\mathbb{C})$. This algebra has the peculiar characteristic that it admits two different invariant quadratic forms; in particular, if we denote the six generators of $\mathfrak{sl}(2,\mathbb{C})$ with (J_a, K_a) a=1,2,3 (J_a identified with the rotations generators and K_a with the boosts) it is always possible to choose two representations of these operators such that (see [21]):

$$trace_1 \;\longrightarrow\; tr_1(J_aK_b) \;=\; \delta_{ab};\quad tr_1(J_aJ_b) = tr_1(K_aK_b) = 0 \quad (1.2.8)$$

$$trace_2 \;\longrightarrow\; tr_2(J_aK_b) \;=\; 0;\quad tr_2(J_aJ_b) = tr_1(K_aK_b) = \delta_{ab} \quad (1.2.9)$$

with these choices the trace appearing in (1.2.4) can be evaluated using either tr_1 or tr_2. One can add the two terms calcuted using both the representations for the trace and can define a full invariant action of the form[2]

$$S[\,\omega\,,\,e\,] = S_{tr_1}[\,\omega\,,\,e\,] + \frac{1}{\gamma}\,S_{tr_2}[\,\omega\,,\,e\,] \quad (1.2.10)$$

[2] This full invariant action is known as the Palatini–Holst action.

γ being a real parameter. This action gives raise to the same equations of motion derived before if no coupling with matter (in the form of spinor fields) is added (this is because only in this case the equation $(D\ e)\,(e,\omega) = 0$ remains the same).

It is remarkable to notice that, at this point, this is a pure classical field theory on a gauge bundle, and the equivalence between this formulation and the Einstein equation holds because this is a *classical* theory. If we try to quantize the fields it may happen that the second term in (1.2.10) may give raise to a *quantum anomaly*, i.e. many procedures of quantization, different one from another, may arise[3]. We will remark this notable difference in the chapter devoted to the spin foams models of quantum gravity, even if each quantization procedure reduces to the Einstein equation of the gravitational field in the classical limit.

1.3 BF theories

In this section we introduce the formalism of the BF-theories. This description characterizes the action principle for fields, on a n-dimensional carrier manifold M, through the assignement of a pair (B, F) of Lie-algebra valued forms. BF-theories play an important role in General Relativity, as one can see in the following when we will show that the Palatini–Einstein action (1.2.4) can be written as a constrained BF-action, the constraint being the B-field to be expressed as the wedge product of two tetrads fields.

Let us take a Lie-algebra valued $(n-2)$-form, namely B, and define a Lie-algebra valued n-form on M by taking the wedge product

$$B \ \wedge \ F \ . \tag{1.3.1}$$

Assume that the Lie group G of the bundle admits a Lie-algebra $\mathfrak{g}$ that can be equipped with a non-degenerate scalar product of the form $\langle A, B\rangle = tr\,(AB)$,

[3]For example, the Barrett-Crane model versus the EPRL/FK model, see chapter 3.

$A, B \in \mathfrak{g}$; employing this hypothesis[4] we could pair the $\mathfrak{g}$-valued parts in (1.3.1) and get a n-form on M by taking

$$tr\,(B \ \wedge \ F) \ . \tag{1.3.2}$$

The functional

$$S\,[\,\omega\ ,\ B\,] = \int_M \ tr\,(B \ \wedge \ F) \tag{1.3.3}$$

is called the BF–action. By making variations on (1.3.3) we obtain the equations of motion:

$$0 = \delta \ \int_M \ tr\,(B \ \wedge \ F) = \int_M \ tr\,[\,\delta\,(B \ \wedge \ F)\,]$$

because of the linearity of the scalar product. So

$$0 = \int_M \ tr\,[\,\delta B \ \wedge \ F + B \ \wedge \ \delta F\,] = \int_M \ tr\,[\,\delta B \ \wedge \ F + B \ \wedge \ \delta \ D\omega\,]$$

we assume $\delta \ D\omega \ = D \ \delta\omega$; the exterior derivative is a graded Leibniz operator, so, after integration by parts, we get

$$0 = \int_M \ tr\,\big[\ \delta B \ \wedge \ F + (-1)^{n-1} \ D \ B \ \wedge \ \delta\omega\big] \tag{1.3.4}$$

This variation vanishes for all δB and $\delta\omega$ if and only if we require that

$$F(\omega) \ = \ 0 \tag{1.3.5}$$

$$(D \ B)\,(\omega, B) \ = \ 0 \ . \tag{1.3.6}$$

These equations are the equations of motion for a BF–type theory. It is easy to remark that a BF–theory is only a *topological* theory, because the condition $F = 0$ states that the connection one-form ω is locally flat; as a consequence ω can be locally expressed as $\omega = g^{-1} \ dg$ (see [14]) and so in different charts ω looks locally the same up to gauge transformations. In this sense we can regard a *topological* theory as a theory without any local degree of freedom, so it is a tool specially suited to deal with physical theories in which the field propagates *with flatness* (for example the gravitational field in absence of matter). Notice that the set of equations one derives from a BF-theory are quite similar to the set of equations one derives from the

[4] In general this condition is not required to be true for *any* Lie group G .

Palatini–Einstein action, eq. (1.2.5) and (1.2.6), apart from a fundamental difference. In a topological theory (like the BF one) one has the "dynamical" equation in the form $F(\omega) = 0$, so no degrees of freedom are included; on the contrary, the Palatini–Einstein "dynamical" equation is $e \wedge F(\omega) = 0$, and here the presence of the tetrad field e wedging the curvature form $F(\omega)$ includes the possibility for the metric to be not necessarily flat.
The other equation, namely (1.3.6), provides a relation between the connection one-form ω and B which states that B must be covariantly *compatible* with ω. We usually refer to it as to the *Gauss law*; if this relation can be solved, one can rearrange equation (1.3.6) in terms of ω.

We want to remark again the difference between a purely topological theory, like the one provided by a BF-action, and the theory of the gravitational field provided by the Einstein–Hilbert action (or, equivalently[5], by the Palatini–Einstein action); in order to stress this fact we use to call

$$F(\omega) = 0 \quad \text{topological eq. with no degrees of freedom} \tag{1.3.7}$$

$$e \wedge F(\omega) = 0 \quad \text{dynamical eq: the metric may be non-flat} \tag{1.3.8}$$

The two solutions may coincide if and only if one introduces further hypotheses on the form of the B field; in particular it must be constrained to be expressed as the wedge product of two one-forms, in this case the tetrads field. Thus, for the action (1.2.10) to be derived from a generical BF-theory, one must introduce a constraint on the B field, for instance in the form of a further field $\phi(B)$ to be added. In particular we write

$$S[\,\omega\,,\ B\,,\ \phi\,] = \int_M tr\,(B\ \wedge\ F) + \phi(B) \tag{1.3.9}$$

where another field ϕ has been introduced in order that its equation of motion take into account the condition for B to be written as the wedge product of tetrads in order to give back the (1.2.10). In this way the gravitational field can be described as a constrained BF-theory (*constrained* because of the presence of the field ϕ which have to take into account the relation between B and the tetrads). Recall that this formulation is really equivalent to the Einstein-Hilbert action only if the equations derived by making variations can be solved exactly and substituted back into the BF-action (namely we

[5] In the sense explained in the previous section.

must be able to solve the Gauss law $D\ B = 0$ and the constraint equation for ϕ).

1.4 Lattice theories

The BF-action provides us an useful tool to write a constrained theory starting from the connection one-form ω (in particular we take its covariant exterior derivative $F = D\ \omega$) and wedging it with a Lie-algebra valued $(n-2)$-form B. Then we impose contraints on the action in the form of a further field $\phi\,(B)$ in order to take into account the relation between ω and the tetrad fields. Now we want to show how a gauge field action[6] can be constructed starting from a lattice decomposition of the space-time and then taking a "continuum limit" in the sense that we are going to specify.
Let us begin by defining the simplest lattice in four dimensions, a 4-hypercube lattice with equal spacing a in each direction. Each point of the lattice is referred to be a *site* and we could denote it by using a number $n \in \mathbb{N}$; for example we take the site called (n) in the space-time. Because we are in four dimensions, each site is linked to other four sites, namely we can call each of them $\left(n + \underline{e}_\mu\right)$, $\underline{e}_\mu$ being the μ^{th} direction in the space-time. Between two neighboring sites of the lattice we introduce a copy of the gauge group G of the theory taking an element $U \in G$. We write this element $U \in G$ as the exponential of a Lie-algebra element which we choose to be the components of the connection one-form ω; as an example, the element between the neighboring sites (n) and $\left(n + \underline{e}_\mu\right)$ can be written in the form

$$U_{n,n+\underline{e}_\mu} \stackrel{def}{=} e^{\ ag\ \omega_\mu(n)} \tag{1.4.1}$$

where $\omega_\mu = \omega_\mu^\alpha\ \underline{e}_\alpha$ is an element of the Lie-algebra $\mathfrak{g}$ of G, a is the lattice spacing and g is a coupling constant.

We now define a plaquette p as a square face of the lattice with dimensions $a \times a$; to each plaquette p we associate the element $U_p \in G$ obtained by taking

[6] in the sense of $S = \int d^4x\ tr\ (\hat{G}_{\mu\nu}\ \hat{G}^{\mu\nu})$.

the product of the elements $U_{n,n+\underline{e}_\mu}$ attached to each pair of neighboring sites of the plaquette. For example a plaquette in the xy plane defines an element

$$U_{\ p^{xy}} = U_{n,n+\underline{e}_1}\ U_{n+\underline{e}_1,n+\underline{e}_1+\underline{e}_2}\ U_{n+\underline{e}_1+\underline{e}_2,n+\underline{e}_2}\ U_{n+\underline{e}_2,n}\ . \tag{1.4.2}$$

A gauge transformation on the element $U \in G$ is defined to be:

$$U_{n,n+\underline{e}_\mu}\ \rightarrow\ \left(U_{n,n+\underline{e}_\mu}\right)' = \Omega\left(n\right)\ U_{n,n+\underline{e}_\mu}\ \Omega\left(n, n+\underline{e}_\mu\right)^{-1} \tag{1.4.3}$$

$\Omega\left(n\right)$ being complex functions. According to this definition the trace of the element U_p associated to each plaquette is indeed gauge invariant, in fact, if we perform gauge transformations on each factor in the definition of U_p, every $\Omega\left(n\right)^{-1}$ cancels against the others in the product. For the plaquette in the xy plane we have

$$\begin{aligned} tr\left(U_{p^{xy}}\right)' \ &= \ tr\left(\ \Omega(n)U_{n,n+\underline{e}_1}\Omega(n+\underline{e}_1)^{-1}\ \ \Omega(n+\underline{e}_1)\times\right. \\ &\times\ \ \left. U_{n+\underline{e}_1,n+\underline{e}_1+\underline{e}_2}\Omega(n+\underline{e}_1+\underline{e}_2)^{-1}\times\ldots\times\Omega(n)^{-1}\ \right) \end{aligned} \tag{1.4.4}$$

so that we have $tr\left(U_{p^{xy}}\right)' = tr\ U_{p^{xy}}$. This enables us to write a gauge invariant candidate for the action by taking

$$S\left[\ \omega\ ; a\right] = -\frac{1}{2g^2}\sum_p tr\ U_p\ . \tag{1.4.5}$$

Now we have to prove that if we take the continuum limit on (1.4.5) in the sense of

$$\lim_{a\to 0}\ S\left[\ \omega\ ; a\ \right] \tag{1.4.6}$$

we can reproduce the gauge field action. Each term in (1.4.5) is in the form of an exponential of Lie-algebra elements, so products of such factors do appear. In order to evaluate this expression we make use of the Baker-Campbell-Hausdorff formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\ldots}\ . \tag{1.4.7}$$

In general we would have an infinite number of commutators; anyway, because we want to take the limit $a \to 0$ we are only interested in the first order terms in this expansion. The lowest order terms are of the form

$$e^{\ ag\ \left[\ \omega_\mu\left(n+\underline{e}_\rho\right)-\omega_\mu(n)\ +\ commutators\ \right]} \tag{1.4.8}$$

where the difference $\omega_\mu\left(n+\underline{e}_\rho\right)-\omega_\mu\left(n\right)$ will become a derivative in the ρ^{th} direction when we take the continuum limit. Putting everything together the action can be written as [7]

$$S\left[\ \omega\ ;a\right]=-\frac{1}{2g^2}\sum_p tr\ e^{\ a^2g^2\ G_{\mu\nu}(n)+...} \tag{1.4.9}$$

where $G_{\mu\nu}(n)=\left(D\ \omega\right)_{\mu\nu}=\partial_\mu\omega_\nu(n)-\partial_\nu\omega_\mu(n)-g\left[A_\mu(n),A_\nu(n)\right]$ are the components of the field strenght. We now expand the exponential and take the limit $a\to 0$; in this way we find the continuum result

$$S\left[\ G\ \right]=-\frac{1}{2g^2}\sum_p\left(1-\frac{a^4}{2}g^2\ tr\left(G_{\mu\nu}G^{\mu\nu}\right)\right) \tag{1.4.10}$$

which becomes, in the continuum interpretation

$$S\left[\ G\ \right]=constant+\int d^4x\ \frac{1}{4}\ tr\left(G_{\mu\nu}G^{\mu\nu}\right) \tag{1.4.11}$$

i.e. the free gauge field action for a Yang-Mills theory on a fibre bundle.

1.5 Ashtekar variables

In the previous sections we wrote the energy density associated to the Einstein-Hilbert Lagrangian in terms of the auxiliary variables of the metric $q_{ab}(g)$ (cfr. pag. 17). Now we want to achieve the same result starting from the action (1.2.10), but using a new set of variables.

The carrier manifold M for the field theory has ben chosen in order to satisfy the condition (1.1.3); now we state precisely that M must be of the form $M=\mathbb{R}\ \times\ \Sigma$, where the real line $\mathbb{R}$ represents "time" and Σ is a compact and orientable (in order to define an integration measure) $(n-1)$ dimensional manifold representing "space". This can be achieved without loss of generality (see [3]).
Let us define the *time-gauge* for the tetrads coefficients. According to the

[7] for the explicit calculation see [9] .

splitting (1.1.7) the vector valued part e_0^I of the tetrad e can be put in the form

$$e_0^I = nN^I + U^a\ e_a^I\ . \tag{1.5.1}$$

The time-gauge consists in the choice $e_0^I = N^I = (1,0,0,0)$ everywhere on M. This allows the introduction of the following linear function of the one-form connection

$$A_a^i = \omega_a^i + \gamma\ K_{ab}\ e^{bi} \tag{1.5.2}$$

with $a, i = 1,2,3$ and K_{ab} being the extrinsic curvature defined in (1.1.17). Notice that, because of the choice of the time-gauge, we have reduced the degrees of freedom of the tetrad field, and (1.5.2) is now a $\mathfrak{su}(2)$-valued connection one-form.

Taken the Lagrangian density defined in (1.2.10), one can calculate the conjugate momentum with respect to the variables A_a^i

$$\frac{\partial \mathscr{L}}{\partial \dot{A}_a^i} = e_i^a\ \det(e) \stackrel{def}{=} E_i^a\left(A_a^i,\ \partial\ A_a^i\right) \tag{1.5.3}$$

which in principle might be function of the A's and of its derivatives, but this is not the case. The variables A_a^i and their conjugate momenta E_i^a are referred to be the *Ashtekar variables*. Inverting (1.5.3), the action (1.2.10) can be written in terms of this new kind of variables, in particular we obtain, after some calculations (see [6]):

$$S\left[\ A\ ,\ E\ \right] = \frac{1}{\gamma}\int_M \left[\dot{A}_a^i\ E_i^a - A_0^i\ G_i - nH - U^a\ H_a\right] \tag{1.5.4}$$

where

$$G_i = D_a E_i^a = \partial_a E_i^a + \varepsilon_{kil} A_a^k E^{al} \tag{1.5.5}$$

$$H_a = \frac{1}{\gamma}\ F_{ab}^j E_j^b - \frac{1+\gamma^2}{\gamma}\ K_a^i G_i \tag{1.5.6}$$

$$\begin{aligned} H = {} & \left[F_{ab}^j - \left(\gamma^2+1\right)\varepsilon_{jmn}\ K_a^m\ K_b^n\right]\ \frac{\varepsilon_{jkl}\ E^{ak}\ E^{bl}}{\det E}\ + \\ & + \frac{1+\gamma^2}{\gamma}\ G^i\ \partial_a\ \frac{E_i^a}{\det E}\ . \end{aligned} \tag{1.5.7}$$

The equations of motion one derives by making variations are exactly the (1.2.5) and (1.2.6), provided the substitutions of the tetrads and the curvature in terms of the Ashtekar variables, taking into account the choice of

the time-gauge which reduces the degrees of freedom. On the dynamical solutions, i.e. "on-shell", one obtains the constraints

$$G_i = 0; \qquad H_a = 0; \qquad H = 0. \tag{1.5.8}$$

The equation $G_i = 0$ is nothing but the requirement $D\,E = 0$, i.e. the Gauss law, which provides local gauge invariance for the E-form. In the same spirit of (1.1.22) we regard (n, U^a) as Lagrange multipliers and the constraints derived by making variations are $H = 0$ and $H_a = 0$. It is easy to convince that even in this case, H and H_a act as the generators of the space-time diffeomorphism group *Diff* (M) for the action (1.2.10).

The classical configuration space for such a description is the space Q of all connection forms A restricted to a "fixed time" (because we employed the time-gauge), so we can think at it as to the set of connection forms on $M_{fixed\ time} = \{0\} \times \Sigma \sim \Sigma$. The corresponding classical phase space is the cotangent bundle T^*Q; a point in this space consists exactly of a pair (A, E), so one can construct a Hamiltonian theory based on the Palatini–Holst action by taking the Legendre transform on the Lagrangian (1.5.4). Then must be imposed the canonical commutation relations between the conjugate variables A and E in the form

$$\left\{A^i_a(m)\ ,\ E^b_j(m')\right\} \propto \delta^i_j\ \delta^b_a\ \delta^{(3)}\left(\ \phi(m)_{|\Sigma} - \phi(m')_{|\Sigma}\ \right) \tag{1.5.9}$$

where (U, ϕ) is a chart containing m; the Poisson brackets bewteen two A's and two E's obviusly vanish everywhere on M. The equations of motion for the fields A and E are derived by considering the canonical Poisson brackets with the Hamiltonian [8]

$$\dot{A}^i_a = \left\{H(A,E)\ ,\ A^i_a\right\} \quad \text{and} \quad \dot{E^b_j} = \left\{H(A,E)\ ,\ E^b_j\right\}\ . \tag{1.5.10}$$

Indeed the solutions of these equations, if manipulated, give back the Einstein equations of motion in terms of the tetrads coefficients (and, as a consequence, in terms of the metric).

[8] The time derivative is to be intended as acting on the representation of a point m in M, namely on $A^i_a\left(\phi(m)\right)$.

1.5.1 Holonomies and fluxes

At this point we have achieved is the description of the classical theory of the gravitational field implemented as a gauge theory on a SU(2)-bundle. We have introduced a new pair of variables, namely the Ashtekar variables, and we have just written the action and the Hamiltonian in terms of these new variables. Now the next step is to construct a set of variables taking into account the topology of the space-time M. An intuitive way for investigating the structure of a manifold is by studying the holonomies. Consider the following example: if we are on a two-dimensional flat Riemannian manifold, a closed circle around a point surrounds an angle of 2π; on the contrary, if the manifold is inherently curved, the full angle enclosed is no longer 2π, but it can be greater or smaller depending according to whether the manifold is locally convex or concave. In this way, by assigning an angle, i.e. an element of the group U(1), to each closed curve surrounding a point, one can take into account the intrinsic curvature of the manifold. Furthermore, by taking any sort of loop on a manifold you can match the topological properties in terms of connectivity by contracting each loop (if possible) to a point; furthermore holonomies can be constructed also taking a sort of "regularization" of the manifold M, i.e. one can cover the entire manifold with a lattice and evaluate the holonomies on this discrete scheme of points. Indeed this method also provides us a well defined procedure of quantization for the pure metric theory of the space-time, this being obtained through the assignement of an element of the gauge group G to each loop (in the following we use the term *triangulation* of the space-time) and following the scheme shown in the section § 1.4 of lattice gauge theories. Morally speaking, a copy of the gauge group G is "attached" to each chosen discretization of the carrier manifold M in order to reconstruct a sort of Yang-Mills action for the gauge field (cfr. § 1.4); this is indeed the procedure we try to achieve.

The Ashtekar variables introduced in the previous section can be regarded as the coefficients of vector-valued one-forms on Σ; in particular consider

$$A = A_a^i \, dx^a \otimes \sigma_i \quad \text{and} \quad E_j^b \, dx^j \otimes \sigma_j \tag{1.5.11}$$

where σ_i are SU (2) generators. Notice that the Ashtekar variables are functions on a 3-dimensional manifold Σ embedded in M, because we fixed the time-gauge. As a consequence, the Lie group G of this *embedded* bundle is no

longer the group SO(3, 1) (or its covering SL(2, $\mathbb{C}$)) but it is now the group of spatial rotations SO(3), or its covering SU(2).

Let c be a path on Σ parameterized by

$$\begin{aligned} c : [0,1] &\longrightarrow \Sigma \\ s &\longrightarrow m(s) \ . \end{aligned} \tag{1.5.12}$$

Given a connection one-form A we can associate to it an element in the Lie-algebra $\mathfrak{su}(2)$ by taking

$$\int_c A = \int_0^1 ds \ A\left[\phi(m(s))\right] \tag{1.5.13}$$

this element can be exponentiated in order to give back an element of the group SU(2). In particular we define the *holonomy* on the path c of the one-form A to be:

$$h_c\left(A\right) = \mathcal{P}\ \left(e^{\int_c A}\right) \tag{1.5.14}$$

where the notation means that the exponential is defined by the path ordered series, namely

$$h_c\left(A\right) \stackrel{def}{=} \sum_{n=0}^{\infty} \int \cdots \int_{1>s_1>\ldots>s_n>0} ds_1 \ldots ds_n \ A\left[c(s_1)\right] \ldots A\left[c(s_n)\right] \ . \tag{1.5.15}$$

Notice that this series expansion coincides with the "time-ordered" product expansion for the time evolution operator of a quantum mechanical system, provided the ordering to be "path-ordered" instead of "time-ordered". This means that $h_c(A)$ is a solution of the same differential equation solved by the time-evolution operator in quantum mechanics

$$\frac{d}{dt}\ h_c(s) = A\left[\ c(s)\ \right]\ h_c(s); \qquad h_c(0) = 1 \tag{1.5.16}$$

If we take a set of paths which meet only at the endpoints in order to furnish a closed curve piecewise continous, we can extend the definition of the integral (1.5.14) to a closed curve γ on Σ, obtaining in this way the holonomy on a loop.

Consider now the vector-valued one-form E and take the vector-valued 2-form on Σ defined by $*E$. This 2-form can be integrated on a surface on Σ

(for example we can take the surface whose boundary is the closed curve γ on which we integrated the one-form A) in order to obtain an element of the Lie-algebra $\mathfrak{su}(2)$

$$\mathcal{F}_S(E) = \int_{S:\ \partial S=\gamma} *E\ . \tag{1.5.17}$$

We refer to $\mathcal{F}_S(E)$ as to the *flux* of E across S. Taking into account the canonical Poisson brackets (1.5.9) one can also calculate the Poisson brackets between the holonomies and the fluxes. It is easy to check that:

$$\{h_c(A)\ ,\ h_c(A')\} = 0 \tag{1.5.18}$$

$$\Big\{(\ \mathcal{F}_S(E)\ )^j\ ,\ (\ \mathcal{F}_S(E)\ )^k\Big\} = \epsilon_i^{\ jk}\,(\ \mathcal{F}_S(E)\ )^i \tag{1.5.19}$$

$$\Big\{(\ \mathcal{F}_S(E)\ )_j\ ,\ h_c(A)\Big\} = \pm\, h_c(A)\ \underline{e}_j \tag{1.5.20}$$

where $\underline{e}_j$ is the basis of the Lie-algebra $\mathfrak{g}$ on which A and E are valued and $(\ \mathcal{F}_S(E)\ )^j$ is intended to be the j^{th} component of the vector valued part of the flux. Equation (1.5.20) vanishes if the curve c does not intersect the surface S; if it does, the $\pm$ sign has to be chosen according to the orientation of c with respect to S.

1.5.2 A classical example: the rigid rotator

The holonomies and the fluxes, introduced out of the Ashtekar variables in the previous section, are not a prerogative of the description of the gravitational field; in fact now we show that there exist a classical system, namely the rigid rotator, whose dynamical description on the phase space is obtained with the same Poisson structure holding for the holonomies and the fluxes. The configuration space Q for a rigid rotator (a spinning top) is SO(3), because each of its positions can be obtained by giving an orientation, which can be obtained by a unique rotation. Taken a chart (U_i, ϕ_i) on SO(3) a set of local coordinates can be given by the Euler angles $\{\alpha^i\} = \{\alpha, \beta, \gamma\}$ which identify a specific rotation $g \in \mathrm{SO}(3)$.

Every smooth function $f \in \mathscr{F}(G)$ defines a configuration variable, and any vector field $X \in \mathfrak{X}(G)$ a momentum variable: $P_X \stackrel{def}{=} X^i p_i$ on $T^*(G)$.

This procedure automatically induces a Poisson structure on $T^*(G)$ in the following way:

$$\begin{aligned} \{P_X, P_Y\} &= P_{[X,Y]} \\ \{P_X, f\} &= L_X\, f \\ \{f_1, f_2\} &= 0 \end{aligned}$$

Now, according to which vector field $X \in \mathfrak{X}(G)$ we choose, a different Poisson structure arises. A natural choice is to take as vector fields defining the Poisson structure, the set of left invariant vector fields $\in \mathfrak{X}(G)$ which parallelize the Lie group. In the case of $G = \mathrm{SO}(3)$ this set is given by the angular momenta X_i such that, in any point $g \in G$:

$$[X_i(g)\ ,\ X_j(g)] = \epsilon_{ij}{}^{k}\ X_k(g)\ . \tag{1.5.21}$$

Taken a chart (U_i, ϕ_i) on G, whose set of local coordinates is $\{x^j\}$, the representation of the angular momenta in this chart is given by $X_i = \epsilon_{ij}{}^{k}\ x^j\ \underline{e}_k$. In this way, the momentum functions associated to such vector fields admit the familiar expression: $P_{X_i} = \epsilon_{ij}{}^{k}\ x^j\ p^k \equiv l_i$. The Poisson structure then becomes:

$$\{P_{X_i}, P_{X_j}\} = \{l_i, l_j\} = P_{[X_i,X_j]} = P_{\epsilon_{ij}{}^{k} X_k} = \epsilon_{ij}{}^{k}\ l_k \tag{1.5.22}$$

$$\{P_{X_i}, f\} = \{l_i, f\} = L_{X_i}\ f = X_i[f] = \epsilon_{ij}{}^{k}\ x^j\ \partial_k f \tag{1.5.23}$$

$$\{f_1, f_2\} = 0 \tag{1.5.24}$$

Notice the parallelism between these Poisson brackets and the Poisson brackets given in the previous section between holonomies and fluxes. The functions $f \in \mathfrak{X}(G)$ are the analogue of the holonomies, and thus their Poisson brackets always vanish, as in (1.5.18). The momentum functions l_i are the analogue of the fluxes, because their action on a function $f \in \mathfrak{X}(G)$ (i.e. on a "holonomy") rotates the function itself ($\epsilon_{ij}{}^{k}\ x^j\ \partial_k f$ can be interpreted as the rotation of the function f along the k^{th} axis), in the same way as the induced flux $\mathcal{F}_S(E)$ rotates the holonomy in equation (1.5.20). Poisson brackets between angular momenta, furthermore, mirror (1.5.19) with the same structure constants $\epsilon_{ij}{}^{k}$.

The Hamiltonian for a rigid rotator is

$$\mathscr{H} = \frac{1}{2}\ \mathrm{I}\ l_k l^k \tag{1.5.25}$$

and enables us to find the equations of motion in the form:

$$\dot{l}_i = \{\mathscr{H}, l_i\} \propto \left\{l_k l^k, l_i\right\} = 0 \tag{1.5.26}$$

$$\begin{aligned}\dot{g} &= \{\mathscr{H}, g\} = \frac{1}{2}\,\mathrm{I}\; l^k \{l_k, g\} = \frac{1}{2}\,\mathrm{I}\; l^k\; L_{X_k}\; g = \\ &= \frac{1}{2}\,\mathrm{I}\; l^k\; X_k\,[g] = \frac{1}{2}\,\mathrm{I}\; l^k\; \epsilon_{kr}{}^s\; x^r\; \partial_s g\end{aligned} \tag{1.5.27}$$

Notice that the Poisson bracket $\{l_k, g\} = X_k\,[g]$ appears in the same form of equation (1.5.20); in fact if g is expressed as $g = e^{i\; c^j X_j}, X_j \in \mathfrak{X}(G)$, then

$$X_k\,[g] = g\; X_k\left[i\; c^j X_j\right] = gi\; c^j\; X_k\,[X_j] = gi\; c^j\; \delta_{jk} = gi\; c_k \tag{1.5.28}$$

which is the same expression shown in (1.5.20), provided the identification of c^k with the k^{th} direction in the Lie-algebra $\mathfrak{so}(3)$. Using this formula, equation (1.5.27) becomes

$$\dot{g} = \frac{1}{2}\,\mathrm{I}\; l^k\; gi\; c_k\;. \tag{1.5.29}$$

Let us introduce the variable $g^{-1}\dot{g}$, then we have, as a consequence:

$$g^{-1}\dot{g} = \frac{1}{2}\,\mathrm{I}\; l^k i\; c_k \tag{1.5.30}$$

which is the common equation of motion for a classical rigid rotator.

Chapter 2

The space of quantum states: spin networks

"Spin networks were first introduced as a radical, purely combinatorial description of the geometry of spacetime. In their original form, they were trivalent graphs with edges labelled by spins. It thus came as a delightful surprise the discovery that spin networks can be used to describe states in loop quantum gravity."

J.C. Baez

Once one has achieved the classical field theory of the gravitational field, the problem of quantization arises. In this section we construct the space of quantum states for the theory starting from a lattice decomposition of the carrier space, following the prescription shown in [3]; in this way one can take into account the topology of the space-time. This space of quantum states must be shown to be a Hilbert space, even if we take into account the constraints imposed from the theory.
Subsequently relevant operators have to be defined acting on these particular stases (which are *not* in space-time, but which *are* the space-time), in order to have a well defined procedure of quantization from which one can also check the classical limits.

2.1 Regularization of the space-time: quantum states

The starting point for the definition of the space of quantum states is the Hamiltonian description of the field theory in terms of the canonical variables A and E, which are viewed as vector valued one-forms whose components satisfy the Poisson brackets relations (1.5.9):

$$\left\{A_a^i(m)\ ,\ E_j^b(m')\right\} \propto \delta_j^i\ \delta_a^b\ \delta^{(3)}\left(\ \phi(m)_{|\Sigma} - \phi(m')_{|\Sigma}\ \right) \tag{2.1.1}$$

the other Poisson brackets vanishing. We can regard a pair (A, E) as a point in the phase space T^*Q, Q being the dynamical configuration space of all the connection one-forms A. The procedure of quantization must turn T^*Q into a Hilbert space of functions $L^2(Q)$; after that, constraints must be imposed on the states. The first constraint we want to introduce in the theory is the gauge invariance, so that the "constrained" phase space $T^*(Q/G)$ must be turned into a "constrained" Hilbert space of functions $L^2(Q/G)$. The notation (Q/G) should be intended as "the configuration space Q constrained by means of a group of transformations G". This purpose is summarized by the following diagram:

$$\begin{array}{ccc} T^*Q & \xrightarrow{quantization} & L^2(Q) \\ {\scriptstyle constraints}\downarrow & & \downarrow{\scriptstyle constraints} \\ T^*(Q/G) & \xrightarrow{quantization} & L^2(Q/G) \end{array} \tag{2.1.2}$$

Remark that this diagram is at present known as following only the direction $T^*Q \overset{quantization}{\longrightarrow} L^2(Q) \overset{constraints}{\longrightarrow} L^2(Q/G)$. The other direction, namely the one passing first through the imposition of the constraints on the phase space and then to the quantization is not yet achieved, and it is a difficult point to solve.

Now we show how to take a lattice regularization of the space-time in order to construct the quantum states.

Consider a finite collection of paths $c_i : [0,1] \longrightarrow \Sigma$ which intersect only at their endpoints. Each path in the collection will be called an *edge* and each endpoint a *vertex*. A path is said to be *incoming* to a vertex v if $c_i(1) = v$; on

the contrary such path is said to be *outcoming* from the vertex v if $c_i(0) = v$. We want this set of paths to be oriented with respect to the endpoints, so we take the collection such that there is a certain number of edges incoming to a vertex and a certain number of edges outcoming from it. If this set of paths $\{c_i\}$ form a closed loop, we call it a graph Γ_n constructed out of n vertices and n paths $c_i,\ i = 1, \ldots, n$.
Such graph is nothing but a piecewise continuous curve on the manifold Σ; in this way one car recover the entire space-time taking a lot of these graphs Γ_n and varying the number n of vertices and edges. We refer to this procedure as to the *regularization* of the space-time obtained by taking a set $\{\Gamma_n\}$ of graphs on Σ.

Consider a graph Γ_n and let γ_k be one of its n edges. Take the connection one-form A and define the holonomy on the edge γ_k as

$$h_{\gamma_k}(A) = \mathcal{P}\ \left(e^{\int_{\gamma_k} A}\right) \quad . \tag{2.1.3}$$

This is an element of the gauge group $G = SU(2)$[1] depending strictly on the one-form A and on the edge γ_k . Any pair (Γ_n, f), where Γ_n is a graph and f is a function of the form [2]

$$\begin{aligned} f:\ SU(2)^n &\longrightarrow \mathbb{C} \\ (h_{\gamma_1}, \ldots, h_{\gamma_n}) &\longrightarrow f(h_{\gamma_1}, \ldots, h_{\gamma_n}) \end{aligned} \tag{2.1.4}$$

is called a cylindrical function, so $(\Gamma_n, f) \in Cyl(\ SU(2)^n\ )$. The value assumed by this function in a point $(h_{\gamma_1}, \ldots, h_{\gamma_n}) \in SU(2)^n$ is a complex number which we refer to by introducing the notation $\Psi_{\Gamma_n, f}(A)$ because it depends on

- the graph Γ_n through its edges γ_n
- the function f chosen in (2.1.4)
- the one-form A integrated on the edges γ_n .

A useful notation is $\Psi_{\Gamma_n, f}(A) = \langle A|\,(\Gamma_n, f)\rangle$ stating that the cylindrical function becomes a complex number when contracted (in the sense of "evaluated") on the one-form A. This notation reminds us the representation of

[1] From now on we intend the gauge group G to be SU(2) .
[2] We sometimes relax the notation and do not indicate that $h_{\gamma_k} \equiv h_{\gamma_k}(A)$.

a quantum state as a wave function.
The space of cylindrical functions on a given graph Γ_n can be turned into a Hilbert space by introducing a scalar product. In order to do this we need a gauge invariant measure on SU(2). Consider the vector valued Maurer-Cartan one-form $g^{-1}\ dg$, g being obviously an element of SU(2); this form is indeed left invariant. Take

$$g^{-1}\ dg\ \wedge\ g^{-1}\ dg\ \wedge\ g^{-1}\ dg \tag{2.1.5}$$

and compute $tr\,(\ g^{-1}\ dg\ \wedge\ g^{-1}\ dg\ \wedge\ g^{-1}\ dg\ )$. Because of the cyclicity of the trace, this turns out to be a gauge invariant 3-form on SU(2), which can be used as a volume form for a scalar product. The key point is that (2.1.5) is the only gauge invariant volume form on SU(2), namely the Haar measure. A well defined scalar product is constructed by setting:

$$\langle (\Gamma_n, f)\,|\,(\Gamma_n, g)\rangle = \int_{SU(2)^n} d\mu\ \ f^* g \tag{2.1.6}$$

$d\mu$ denoting the Haar measure. If the space of cylindrical functions is not complete according to the Haar measure we can complete it by adding its own closure and define an Hilbert space $\mathcal{H}_{\Gamma_n}$ on the graph Γ_n. The next step is to define a "total" Hilbert space for all cylindrical functions and all graphs. If $\{\Gamma_n\}_{n\in\mathbb{N}_0}$ is a recovering of Σ obtained by taking all the possible graphs varying the edges and the vertices, a natural choice is to take as "total" Hilbert space the direct sum of Hilbert spaces on each graph

$$\mathcal{H}_{total} = \bigoplus_{\Gamma_n\subset\Sigma} \mathcal{H}_{\Gamma_n}\ . \tag{2.1.7}$$

The scalar product on $\mathcal{H}_{total}$ is easily induced from the scalar product on each $\mathcal{H}_{\Gamma_n}$. An element of $\mathcal{H}_{total}$ is a linear combination of elements in each $\mathcal{H}_{\Gamma_n}$ (because of the definition of direct sum). As a consequence, the scalar product between elements of $\mathcal{H}_{total}$ is well defined once we define the scalar products bewteen elements of two different $\mathcal{H}_{\Gamma_n}$. Let $|(\Gamma_n^{(1)}, f)\rangle$ and $|(\Gamma_k^{(2)}, g)\rangle$ be two cylindrical functions (indeed two *states*). If $\Gamma_n^{(1)} = \Gamma_k^{(2)}$ then (2.1.6) immediately applies; otherwise, if $\Gamma_n^{(1)} \neq \Gamma_k^{(2)}$, we consider a further graph obtained by taking $\Gamma_{n,k}^{(3)} = \Gamma_n^{(1)} \cup \Gamma_k^{(2)}$ and extend the definition of f and g on $\Gamma_{n,k}^{(3)}$ by taking $\tilde{f}$ and $\tilde{g}$ which act as f and g on the subset of $\Gamma_{n,k}^{(3)}$ they have in common, and trivially otherwise. In this way the scalar product between $|(\Gamma_n^{(1)}, f)\rangle$ and $|(\Gamma_k^{(2)}, g)\rangle$ reads:

$$\langle\ (\Gamma_n^{(1)}, f)\ |\ (\Gamma_k^{(2)}, g)\ \rangle \stackrel{def}{=} \langle\ (\Gamma_{n,k}^{(3)}, \tilde{f})\ |\ (\Gamma_{n,k}^{(3)}, \tilde{g})\ \rangle\ . \tag{2.1.8}$$

The key result is that this extended scalar product is equivalent to a scalar product in the total Hilbert space (2.1.7) taken with rispect to a particular extended measure, namely the Ashetakar-Lewandowski measure $d\mu_{AL}$. This means that a measure $d\mu_{AL}$ does exist such that:

$$\langle\ (\Gamma_n^{(1)}, f)\ |\ (\Gamma_k^{(2)}, g)\ \rangle = \int_{SU(2)^n} d\mu_{AL}\quad f^* g \tag{2.1.9}$$

indeed satisfies the requirements for a scalar product, and so $\mathcal{H}_{total}$ is really a Hilbert space of quantum states. The mathematical framework underlying the definition of this measure is indeed full of technical details, so we refer the reader to the original paper, namely [1], for the analysis of these further details. Notice that this space is non-separable, i.e. it doesn't admit a countable basis. Therefore, most of the spectral theorems of linear operators on an Hilbert space cannot work in this case.

This procedure completes the first step in the diagram (2.1.2). The Hilbert space of quantum states is the set of all cylindrical functions on all graphs, equipped with the scalar product defined by the Ashtakar-Lewandowski measure $d\mu_{AL}$. Now constraints must be imposed on this Hilbert space in order to obtain a space of states invariant under gauge transformations. To this aim we define a new class of states, the *spin networks*, and show that the Hilbert space they span is indeed invariant under gauge transformations.

Definition. *A "spin network" in* Σ *is a triple* $\Psi = (\ \Gamma_n, \{\rho_r\}, \{\tau_n\}\ )$ *consisting of:*

1. *a graph* $\Gamma_n \subset \Sigma$.
2. *a set* $\{\rho_r\}$ *of representations of the gauge group* G *for each edge* γ_r *of* Γ_n.
3. *a set* $\{\tau_n\}$ *of operators (called "intertwiners") for each vertex* v *of* Γ_n *in the form:*

$$\tau_v : \rho_{\gamma_1} \otimes \ldots \otimes \rho_{\gamma_n} \longrightarrow \rho_{\gamma'_1} \otimes \ldots \otimes \rho_{\gamma'_m} \tag{2.1.10}$$

where $\gamma_1, \ldots, \gamma_n$ *are the edges incoming to* v *and* $\gamma'_1, \ldots, \gamma'_m$ *are the edges outcoming from* v.

Given this definition, we have to show that it is possible to get a set of gauge invariant functions out of a spin newtork which can be turned into a Hilbert space itself gauge invariant. The idea is quite simple. Consider the holonomy h_{γ_r} defined integrating the one-form A on the edge γ_r; this is an element of the group SU(2); on the edge of integration the spin network $\Psi = (\ \Gamma_n, \{\rho_r\}, \{\tau_n\}\ )$ provides a representation ρ_r of the group SU(2), so we have a matrix representing the holonomy h_{γ_r} whose entry (i, j) is

$$[(\rho_r)\,(h_{\gamma_r})]^i_j \tag{2.1.11}$$

We evaluate these coefficients for each edge in Γ_n, getting an entry labelled by different indices for each edge. Then we use the intertwiner operators to contract each upper index with an appropriate lower index of the corrisponding representation of the holonomy. Because of the orientation chosen for the edges, we could define the upper or lower position of an index according to whether the edge is incoming to the vertex v or outcoming from it. Notice that the role played by the intertwiner operators is to "glue together" the representation of the gauge group at each vertex.

A simple example can clarify the ideas. Let us take the spin network Ψ with 2 vertices and 3 edges shown in *Fig.* 2.1. Let us introduce the convention such that the row index in the matrix representation labels the edge outcoming from the vertex v, while the column index labels the edge incomint to it. To the edge e_1 is associated the element $[(\rho_1)\,(h_{e_1})]^a_b$; to the edge e_2 the element $[(\rho_2)\,(h_{e_2})]^c_d$ and to the edge e_3 the element $[(\rho_3)\,(h_{e_3})]^e_f$. These indices must be contracted with the indices of the intertwiner tensors, so we get the following function of the spin network Ψ and of the connection one-form A

$$\Psi\,(A) = [(\rho_1)\,(h_{e_1})]^a_b\ \ [(\rho_2)\,(h_{e_2})]^c_d\ \ [(\rho_3)\,(h_{e_3})]^e_f\ \ (\tau_{v_1})_{ace}\ \ (\tau_{v_2})^{bdf} \tag{2.1.12}$$

The element $\Psi\,(A)$ can be regarded as the image of a function $\Psi : Q \longrightarrow \mathbb{C}$ on the configuration space of all the connection one-forms. Using the fact that the operators τ_n are intertwiners (i.e. they are tensors matching different representation of the gauge group) one can convince that these functions $\Psi : Q \longrightarrow \mathbb{C}$ are indeed gauge invariant, because after performing a gauge transformation in (2.1.12) some terms change "covariantly" (in the sense that they change according to the jacobian of the transformation) and some others change "contravariantly" (in the sense that they change according to the

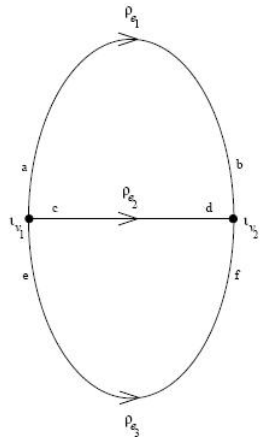

Figure 2.1: *This is a spin network with 2 vertices and 3 edges.*

inverse-jacobian of the transformation) giving a total gauge invariance. The hard part is to prove that the set of these functions can be equipped with a scalar product and can be turned into a Hilbert space by adding its own closure. We do not give a proof of this statement here, and refer the reader to [3] for further details.

The entire set of spin networks $|\Psi\rangle$ thus span the entire gauge invariant Hilbert space, and the "wave function representations" $\Psi(A) = \langle A|\Psi\rangle$ give raise to the set $L^2(Q/G)$ of complex functions on the configuration space Q/G. In this way we have achieved the second step in our quantization scheme of (2.1.2).

At this stage we have to face the problem of the reduction of the gauge invariant configuration space Q/G to a configuration space invariant under both gauge transformations and the diffeomorphism group. We can denote it by means of the more cumbersome but clearer notation $Q/(G \text{ and } \mathit{Diff})$. Then we have to quantize this space in order to get the algebra of complex functions on $Q/(G \text{ and } \mathit{Diff})$ equipped with a scalar product, namely the Hilbert space $L^2(\, Q/(G \text{ and } \mathit{Diff})\,)$. The quantization scheme of (2.1.2) now becomes:

$$\begin{array}{ccc} T^*Q & \xrightarrow{quantization} & L^2(Q) \\ {\scriptstyle constraints}\downarrow & & \downarrow{\scriptstyle constraints} \\ T^*(Q/G) & \xrightarrow{quantization} & L^2(Q/G) \\ {\scriptstyle constraints}\downarrow & & \downarrow{\scriptstyle constraints} \\ T^*(\ Q/(G \text{ and } \mathit{Diff})\ ) & \xrightarrow{quantization} & L^2(\ Q/(G \text{ and } \mathit{Diff})\ ) \end{array} \tag{2.1.13}$$

We remark again that in this scheme one has first to introduce the quantization procedure and only after that constraints can be imposed on the quantum system. The opposite direction is very difficult to achieve, so it will not be point of discussion in this paper.

It turns out to be hard work to implement the last procedure. A diffeomorphism takes a particular graph into another one. We can construct the space of all equivalence classes with respect to such transformations. For this reason we introduce the concept of *topological knot*: a $\mathbb{R}^3$ knot is an embedding s of a circle S^1 into the vector space $\mathbb{R}^3$. Two knots, s and s', are said to be *topologically equivalent* if can be transformed one into another by means of an ambient isotopty of $\mathbb{R}^3$, i.e. if it exists a map

$$F : \mathbb{R}^3 \times [0,1] \longrightarrow \mathbb{R}^3 \tag{2.1.14}$$

such that the embedding s can be turned into s'. This map is a sort of manipulation of the vector space $\mathbb{R}^3$, which can turn a knot into another one. It can be shown that two embeddings which differ for the numbers of knots, regarded as the twisting of the embedding onto itself, cannot be taken one into another by using the map F. For this reason, these knots indeed characterize the equivalence classes of states with respect to the group of diffeomorphisms, because it can be shown that such a transformation can change the way a graph is embedded in Σ, but not the presence of knots within the graph.

We want the set of functions constructed out of these knots to be turned into a Hilbert space. This can be very problematic, because, except for the cases when Σ is compact and of dimension less than 3, and the case when Σ in

simply connected, no natural measure can be constructed in order to get a well defined scalar product; as a consequence, it is not clear whether this set of functions can be be turned into a Hilbert space. It is indeed a very hard question to solve, because a space without a scalar product doesn't satisfy the requirements we need in order to be "physical". Nevertheless, we don't face this question here and consider an element of the algebra of functions on knots as a "physical state". We also relax the notation and call $\tilde{Q}/G$ the configuration space gauge and diffeo-invariant. A physical state $|s\rangle$ is then a knot, and its wave function representation $s(A) = \langle A|s\rangle$ lies within this "maybe non-Hilbert space" $\mathfrak{F}\left(\tilde{Q}/G\right)$ [3].

2.2 Spin foams as evolution of quantum states

The algrebra of functions $\mathfrak{F}(Q_0/G)$ defines the states of the quantum theory. States are built up as equivalence classes of spin networks, each spin network reducing one into another by means of a diffeomorphism on the manifold M. These states constitute indeed the states of the gravitational field, and accordingly, they do not live *somewhere* in the space-time, but they *define* the *where* itself (we can regard them as elementary excitations of the gravitational field). We now want to define a method to obtain the *evolution* between these states in the most general way; thus one can regard this evolution as a transition amplitude bewtween two configurations $|s\rangle$ and $|s'\rangle$[4] of the gravitational field and use this procedure to *define* the action of the Hamiltonian constraint on physical states.

Let $|s\rangle$ and $|s'\rangle$ be two knots, i.e. two equivalence classes of spin networks whose representatives we chose to be Ψ and Ψ'. An *evolution* $|s\rangle \longrightarrow |s'\rangle$ is a set of points in M whose boundaries are Ψ and Ψ'. To this aim we construct

[3] Even if this space cannot bu turned into a Hilbert one, it is a classical results of knots theory that the the entire set of knots admits a countable basis, thus this space is surely separable.

[4] We speak in the same way of *states* and of their *wave function representation* $s(A)$ when there is no possibility of confusion.

a complex κ of dimension 2, that is a topological space obtained by "gluing together" vertices and edges in order to get faces that are 2-dimensional surfaces embedded into M. As we said, this complex is "bounded" by the spin networks Ψ and Ψ' (see *Fig* 2.2).

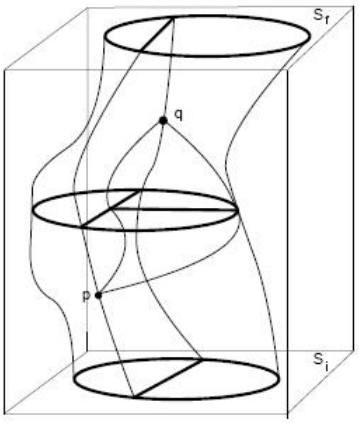

Figure 2.2: *This is a 2-complex κ boundared by Ψ and Ψ', representing the equivalence classes of knots $|s_i\rangle$ and $|s_f\rangle$.*

A 2-dimensional complex κ can be obtained by taking the cartesian product $\Gamma \times I$, where Γ is a one-dimensional complex (for example the graph Γ_n of a spin network) and I is an interval that without loss of generality can be chosen to be $[0, 1]$; so

$$\kappa \stackrel{def}{=} \Gamma \times [0, 1] \ . \tag{2.2.1}$$

If there exists an application $\Gamma \longrightarrow \kappa$ which maps each oriented edge of Γ onto an unique face of κ, we say that Γ borders the complex κ. Equipped with the definition of bordered complexes we can define the evolution between spin networks.

Definition. *Given a spin network $\Psi = (\ \Gamma_n, \{\rho_r\}, \{\tau_n\}\ )$, a type-$\emptyset$ spin foam $F : \emptyset \longrightarrow \Psi$ is a triple $(\ \kappa, \{\tilde{\rho}_r\}, \{\tilde{\tau}_n\}\ )$ consisting of:*

1. *a 2-dimensional oriented [5] complex κ such that Γ_n borders κ.*

[5] A complex is said to be oriented if each faces $f \in \kappa$ is oriented in the same way as its edges.

2. *a set* $\{\tilde{\rho}_r\}$ *of representations of the gauge group* G *for each face* f *of* κ.

3. *a set* $\{\tilde{\tau}_s\}$ *of operators (called "intertwiners") for each edge* e *of* κ *not lying in* Γ_n *in the form:*

$$\tilde{\tau}_e : \tilde{\rho}_{f_1} \otimes \ldots \otimes \tilde{\rho}_{f_n} \longrightarrow \tilde{\rho}_{f'_1} \otimes \ldots \otimes \tilde{\rho}_{f'_m} \tag{2.2.2}$$

where $f_1, \ldots, f_n$ *are the faces incoming to* e *and* $f'_1, \ldots, f'_m$ *are the faces outcoming form* e.

such that the following matching conditions must hold:

- *For any edge* c *of* Γ_n, $\tilde{\rho}_f = \rho_c$ *if* f *is a face incoming to the edge* c*; otherwise it must be* $\tilde{\rho}_f = {\rho_c}^\dagger$.

- *For any vertex* v *of* Γ_n *the intertwiners of the edges of* κ *must be equal to the intertwiners of the spin network* Ψ *in* v *at least up to a dualization of equation* (2.2.2).

The matching conditions are such that if we take a slice embedded in Σ of a type-$\emptyset$ spin foam, we just get a spin network; in this sense we are led to regard a type-$\emptyset$ spin foam as the original spin network "$\times$ an extra dimension". The notation $F : \emptyset \longrightarrow \Psi$ has been introduced because the complex κ is boundered only by the spin network Ψ, so we think of this spin foam as a sort of evolution from the vacuum geometry to the geometry taken into account by Ψ. If we want to evolve a fixed geometry into another one, we should define the spin foam $F : \Psi \longrightarrow \Psi'$; this is achieved by means of the definition of dual and tensor product of different spin networks.

Consider a spin network $\Psi = (\ \Gamma_n, \{\rho_r\}, \{\tau_n\}\ )$, the dual spin network $\Psi^\dagger$ is defined to be the triple with the same underlying graph, but with edges labelled by the adjoint representation $\rho^\dagger$ of the group G and intertwiner operators on each vertex obtained by dualizing the original ones. So

$$\Psi^\dagger = (\ \Gamma_n, \{\rho_r\}, \{\tau_n\}\ )^\dagger = \left(\ \Gamma_n, \left\{\rho_r^\dagger\right\}, \left\{\tau_n^\dagger\right\}\ \right)\ . \tag{2.2.3}$$

Let $\Psi = (\ \Gamma_n, \{\rho_r\}, \{\tau_n\}\ )$ and $\Psi' = (\ \Gamma'_{n'}, \{\rho'_{r'}\}, \{\tau'_{n'}\}\ )$ be two distinct spin networks, their tensor product $\Psi \otimes \Psi'$ is the spin network whose underlying graph is the disjoint union (i.e. same elemenents coming from differents set are not indentified with a unique element) of the original ones, with edges and

vertices labelled by the corresponding representations ρ, ρ' and corresponding intertwiners τ, τ'. Formally

$$\Psi \otimes \Psi' \stackrel{def}{=} \left(\Gamma_n \underset{disjoint}{\cup} \Gamma'_{n'} \ , \ \{\rho_r, \rho'_{r'}\} \ , \ \{\tau_n, \tau'_{n'}\} \right) \ . \tag{2.2.4}$$

The evolution between two quantum states $|s\rangle$ and $|s'\rangle$ (i.e. two geometries) whose representants are Ψ and Ψ' is defined to be:

$$F : \Psi \longrightarrow \Psi' \quad \stackrel{def}{=} \quad F : \emptyset \longrightarrow \Psi^\dagger \otimes \Psi' \ . \tag{2.2.5}$$

Spin foams $F : \Psi \longrightarrow \Psi'$ and $F' : \Psi' \longrightarrow \Psi''$ can be composed into $F \circ F' : \Psi \longrightarrow \Psi''$ by gluing together the two corresponding complexes κ' at Ψ'. Furthermore, we require this composition property to be associative, i.e. $F \circ (G \circ H) = (F \circ G) \circ H$, F, G, H being spin foams for which the composition rule make sense.

2.2.1 Transition amplitudes

Definition. *A spin foam model is an assignment of amplitudes $A[F]$ for each spin foam F such that the following conditions hold:*

1. *$A : F \longrightarrow A[F]$ is a complex valued function.*
2. *$A[F \circ F'] = A[F] \ A[F']$.*

Spin foams take into account the evolution between states of the gravitational field; if we start from a geometry of the space-time described by the knot $|s\rangle$, it evolves into a geometry of the space-time described by the knot $|s'\rangle$ under the action of an evolution operator, we call it $\hat{Z}(M)$ (because it depends on the topological structure of the space-time manifold M), such that the *transition amplitude* between states $|s\rangle$ and $|s'\rangle$ is defined to be:

$$\langle s| \ \hat{Z}(M) \, s' \rangle \stackrel{def}{=} \sum_{F:\Psi \longrightarrow \Psi'} A[F] \tag{2.2.6}$$

Ψ and Ψ' being the representant of the equivalence classes of $|s\rangle$ and $|s'\rangle$. The sum in the above equation implies both a sum over all the possible

2-complexes and a sum over all the possible labelling of elements of them by representations of the group $G = SU(2)$. In this picture spin networks are regarded as states, while spin foam represent *histories* of these states. So equation (2.2.6) is to be intended as a "sum over histories" in the same way as the transition amplitude between states in quantum field theory is computed by calculating the Feynman path integral as a sum over all the possible configurations of the fields. So each spin foam in the sum (2.2.6) is to be regarded as a single Feynman diagram, and the sum over all the possible labelling of elements of a spin foam reproduces the multiple ways we can arrange the legs and the vertices in a Feynman diagram in order to get a scattering amplitude; then, in quantum field theory, one adds up all the contributions coming from each diagram and obtains the perturbative series for a transition amplitude. This is exactly what equation (2.2.6) means.
We also introduce the notation $Z_M(s, s')$ for $\langle s| \hat{Z}(M) s' \rangle$, because we want to think at it as a partition function in the same spirit of the quantum field theory.

The key point is now to find an explicit expression for the quantity $A[F]$ representing the evolution according to a particular spin foam, i.e. a particular Feynman diagram constructed out of a class of elements assigned to it. For example in quantum field theory one picks up vertices, legs and loops, and assigns to each one of them a "Feynman rule", that is a "number", in order to compute the transition amplitude by multiplying all these numbers that label the diagram; in evaluating the total transition amplitude one has to sum up only over all the *topologically distinct* diagrams. In the same way we now introduce a general expression for the amplitude $A[F]$ assigned to a particular spin foam depending only on its constitutive elements (faces, edges, vertices) and then we sum up only over the equivalence classes of spin foams which cannot be obtained one another by simple transformations as

1. *affine transformations*, that is a one-to-one map $\phi : F \to F'$ which assigns faces, edges and vertices of κ to faces, edges and vertices of κ' preserving the orientation.

2. *subdivision*, that is a map $\phi : F \to F'$ which assigns to each face, edge and vertex of κ a fixed number p of faces, edges and vertices of κ' preserving the representations of the group G.

3. *orientation reversal*, that is a map $\phi : F \to F'$ which changes the

orientation of faces and edges of κ dualizing the representation of the group G and the intertwiners.

Let $A_{f_\kappa}^{(J)}, A_{e_\kappa}^{(J)}, A_{v_\kappa}^{(J)}$ be the amplitudes (i.e. the "Feynman rules") assigned respectively to a face, an edge and a vertex of a simplicial complex κ in the representation J of the group $G = \mathrm{SU}(2)$ for the spin foam F. The amplitude of the spin foam is then

$$A[F] = \prod_{faces\ \in\kappa} A_{f_\kappa}^{(J)} \prod_{edges\ \in\kappa} A_{e_\kappa}^{(J)} \prod_{vertices\ \in\kappa} A_{v_\kappa}^{(J)} \quad . \tag{2.2.7}$$

The total transition amplitude of equation (2.2.6) thus becomes, summing over all the representations J and all the complexes κ

$$Z_M(s,s') = \sum_\kappa w(\kappa) \left[\sum_J \left(\prod_{faces\ \in\kappa} A_{f_\kappa}^{(J)} \prod_{edges\ \in\kappa} A_{e_\kappa}^{(J)} \prod_{vertices\ \in\kappa} A_{v_\kappa}^{(J)} \right) \right] \tag{2.2.8}$$

where $w(\kappa)$ is the relative weight we assign to the complex κ in the "perturbative series" (2.2.6). Notice that, even if we have labelled each representation by the SU(2)-like symbol J (indeed the name *spin foam* means *SU(2)-foam*), and the entire sum as a sum over discrete values of J, in the case of non-compact group (for instance the Poincaré group) these representations must be labelled by means of continous indices and the sum over J actually reduces to an integral over the continous parameter labelling the representation.
The crucial difference between quantum field theory and spin foam models of quantum gravity is that the partition functions for spin foams aim to be a non-perturbative series, since it is not based on any kind of perturbative expansion.

In this spirit we want to regard (2.2.8) as to a sort of "Green's function", because, as the Green's functions in quantum field theory carry informations on the propagator, the vertex amplitudes and the interaction among the several components of a field, in the same way expression (2.2.8) carries informations on the transition amplitude between the states s and s', given the space-time manifold M. In the quantum field theory the sum (with suitable weights) of all the Green's functions furnishes the full generating functional, in fact it is

$$Z[J] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \cdots \int d^4x_1 \ldots d^4x_n \; J(x_1) \ldots J(x_n) \; G^{(n)}(x_1 \ldots x_n) \tag{2.2.9}$$

$G^{(n)}(x_1 \dots x_n)$ being the Green's function of order n. This formula represents (up to a normalization constant) the generating functional of the system in presence of an external source J coupling to the field:

$$Z[J] = N \ \langle 0|0\rangle \ . \tag{2.2.10}$$

With the same spirit we want to obtain the transition amplitude between ground states of the gravitational field as a sum over all the possible "gravitational Green's function" $Z_M(s, s')$ with a suitable choice of weights $a(s, s')$

$$\langle 0_{grav}|0_{grav}\rangle \ = Z_{grav} = \sum_{s,s'=knots} a(s, s') \ Z_M(s, s') \ . \tag{2.2.11}$$

There are indeed remarkable differences between the two expressions (2.2.9) and (2.2.11). The former is an exact identity, coming from the power expansion for a functional $Z[J]$; the latter has to be regarded as a *possible choice* for a correct definition of the quantity $\langle 0_{grav}|0_{grav}\rangle$, a sort of hypothesis we argue about the transition between ground states of the gravitational field. This is because in the quantum field theory we do have the full description of the quantum theory itself by hypothesis, in the form of a generating functional for the field; starting from this ingredient we try to obtain some Feynman rules in order to compute scattering amplitudes between states, by using a power expansion in terms of some physical parameters. In the spin foam approach, on the contrary, we *do not* have the definition of what the quantum theory of the gravitational field should be; we start from the Feynamn rules (in the form of spin foams) and try to obtain the generating functional for the field as equation (2.2.11).
Furthermore, equation (2.2.9) contains the coupling of the field with an external source J, which takes into account the possibility for new particles to be created and annihilated, even if no physical information is containet into the current J. In the expression for Z_{grav}, on the contrary, no external source is taken into account, so this expression is indeed quite different from what is referred to be a *generating functional* in the common sense, and the only contribution for the creation and annihilation of new particles must be carried by the self-interaction (if present) of the field itself.

The most important difficulty we have to face in evaluating an expression like (2.2.8) (which represents the single term to be summed over in equation (2.2.11)) is that we do not have any rule to associate a Feynman amplitude

$A^{(J)}$ to each part of a spin foam, neither we have any rule to introduce the weights $w(\kappa)$ for each complex κ; in other words, what is missing is a set of "how-to" rules useful for the explicit calculation of (2.2.8). This is a very relevant difficulty to face and one can be led, in the construction of a particular model, only by criterions of mathematical consistence and group theory motivations in order to achieve a correct formulation for the generating functional of the gravitational field. This is exactly what we try to pursue in the following section, where we are going to introduce two particular models, namely the Barrett-Crane model and the EPRL model. These models are constructed as a path integral quantization starting from a constrained BF-theory, the constraint being the condition for the B field to be simple, i.e. $B = e \wedge e$; each of them tries to implement this condition in a different manner.

Chapter 3

Spin foam models

"When Feynman told me about his version of quantum mechanics, he said that the electron does anything it likes, it goes in any direction at any speed, forward or backward in time, however it likes. Then you have just to add up the amplitudes and it gives you the wave function. I think he was cray, but indeed he wasn't."

F.J. Dyson

In this chapter we regard gravity as a constrained BF-theory. We thus introduce the path integral formulation for a BF-theory using a discretization of the space-time. The transition amplitudes are calculated using the notion of dual skeleton with respect to a particular triangulation taken on the carrier manifold M. We show that in $1+1$ dimensions the calculation is quite simple ([3]); otherwise in $2+1$ and $3+1$ dimensions some mathematical difficulties arise. Furthermore, the condition for the B field to be expressed as the wedge product of two one-forms is not easily discretized. In order to solve these difficulties one is led to impose constraints directly at the level of group representation. This approach allows to compute the partition function and yields to the so-called Barrett-Crane ([3], [17], [8]) and EPRL/FK models ([8]), each of them implementing the simplicity condition for B in a different way.

3.1 Plebanski formulation of the general relativity

The action for general relativity can be introduced, using the tetrad tensor fields, in the form of (1.2.4); the metric tensor is expressed in terms of the tetrad coefficients as $g_{\mu\nu}(m) = e^A_\mu(m) e^B_\nu(m)\ \eta_{AB}$. We have shown, in the previous sections, that the action (1.2.4) is nothing but a particular case of a constrained BF-action in which the B field must be constrained to be the wedge product of two one-forms, in this case the tetrad forms. Using the notation $e = e^I \otimes \underline{v}_I = e^I_\mu dx^\mu \otimes \underline{v}_I$ for the vector valued part of the tetrad field[1] (remember that we are using capital letters for indices raised and lowered by using the Minkowsky metric), the constraints imposed on the B field can appear as a field equation for a Lagrange multiplier ϕ if we define the action

$$S[\ \omega\ ,\ B\ ,\ \phi\ ] = \int_M \left[B^{IJ} \wedge F_{IJ} - \frac{1}{2}\ \phi_{IJKL}\ B^{KL} \wedge B^{IJ} \right] . \qquad (3.1.1)$$

In this case ω is the connection one-form valued in the Lie-algebra $\mathfrak{g}$ of G; B is a Lie-algebra valued 2-form $B = B^{IJ}_{\mu\nu} J_{IJ} dx^\mu \otimes dx^\nu$, with J_{IJ} being the generators of the Lie-algebra $\mathfrak{g}$. The ϕ field appears as a Lagrange multiplier and we demand it to have the symmetries $\phi_{[IJ][KL]} = \phi_{[KL][IJ]}$; furthermore it is required to satisfy $\phi_{IJKL}\ \epsilon^{IJKL} = 0$. This formulation is the so-called Plebanski model of General Relativity.

From now on we reduce our description to the Euclidean case, i.e. we do not assume the gauge group to be $\mathrm{SO}(3,1)$ or its covering $\mathrm{SL}(2,\mathbb{C})$, but the simple rotation group in four dimensions $\mathrm{SO}(4)$.

[1] In the case of $2+1$ dimensions the basis $\underline{v}_I$ can be taken directly in the Lie-algebra $\mathfrak{so}(3)$, because it has dimension 3. In the 4-dimensional case, on the contrary, the Lie-algebra $\mathfrak{so}(4)$ has dimension 6, so it cannot be taken to be in a direct isomorphism with the vector valued part of a tetrad field e.

The equations of motion one derives by making variations are:

$$\begin{cases} \dfrac{\delta S}{\delta \omega} = 0 & \Rightarrow (D\ B)\,(\omega, B) = 0 \\ \dfrac{\delta S}{\delta B} = 0 & \Rightarrow F^{IJ}\,(\omega) = \phi^{IJKL} B_{KL} \\ \dfrac{\delta S}{\delta \phi} = 0 & \Rightarrow B^{IJ} \wedge B^{KL} = e\ \epsilon^{IJKL} \end{cases} \tag{3.1.2}$$

Here $e = 1/4!\ \epsilon_{ABCD}\ B^{AB} \wedge B^{CD}$. When $e \neq 0$, i.e. for non-degenerate metric configurations, the last equation in (3.1.2) implies $\epsilon_{IJKL}\ B^{IJ}_{\mu\nu} B^{KL\ \mu\nu} = 0$. This condition states that B must be simple, that is, it can be expressed as the wedge product of two one-forms. In other words, the requirement $B^{IJ} \wedge B^{KL} = e\ \epsilon^{IJKL}$ is satisfied if and only if there exists a real tetrad field $e^I = e^I_\mu dx^\mu$ such that one of the following conditions hold:

$$I sector \qquad B^{IJ} = \pm\ e^I \wedge e^J \tag{3.1.3}$$

$$II sector \qquad B^{IJ} = \pm\ \frac{1}{2}\ \epsilon_{KL}{}^{IJ}\ e^K \wedge e^L \quad . \tag{3.1.4}$$

These are the expressions, in components, of the equations $B = e \wedge e$ and $B = *\,(e \wedge e)$. According to which solution we choose we get the action

$$S\,[\ \omega\ ,\ e\ ] = \int tr\,[\ e \wedge e \wedge F\ ] \tag{3.1.5}$$

or

$$S\,[\ \omega\ ,\ e\ ] = \int tr\,[\ *\,(e \wedge e)\ \wedge F\ ] \tag{3.1.6}$$

which classically give raise to the same equations of motion, i.e. the Einstein free field equations. So, at the classical level, we can choose arbitrarily one of the two solutions of the constraint equation for B, because it can be proven that initial data into one of the two sectors do not spontaneously evolve into any of the others, provided the tetrads to be non-degenerate. This enables us to write an action which is a linear combination of the two sectors, (3.1.5) and (3.1.6),

$$S\,[\ \omega\ ,\ e\ ] = a_1 \int tr\,[\ e \wedge e \wedge F\ ] + a_2 \int tr\,[\ *\,(e \wedge e)\ \wedge F\ ]\ . \tag{3.1.7}$$

being sure that we obtain the same equations of motions we derive by choosing only one sector out of the two given. Dividing by a_1 or a_2, according to which one of them is $\neq 0$, one can rewrite the above equation in the form

$$S\left[\ \omega\ ,\ e\ \right] = \int tr\left[\ e \wedge e \wedge F\ \right] + \frac{1}{\gamma}\int tr\left[\ *\left(e \wedge e\right)\ \wedge F\ \right] \tag{3.1.8}$$

γ referred to as the Immirzi parameter. This is the form of the action given in equation (1.2.10), where the Hodge–$*$ operator takes into account the trace tr_2 evaluated with respect to a different scalar product in the Lie-algebra of the gauge group $\mathrm{SO}\left(4\right)$ or $SL\left(2,\mathbb{C}\right)$. We obtain, as a result, the so called Palatini-Holst action, which is the starting point for the formulation of general relativity using the Ashtekar variables.

At the quantum level, on the contrary, one cannot arbitrarily choose only one sector of solutions of the constraint equation for B, because interference between both of the sectors cannot be avoided. This fact can be understood very simply if we choose a procedure of quantization using the path integral formulation; in fact, in this way, one has to integrate over all the possible values available of the B field, and so solutions into the two different sectors will interfere.

3.2 Discrete path integral quantization

Quantum field theory provides a general method for implementing the dynamics of a given Lagrangian system (assigned in the form of a Lagrangian density $\mathscr{L} \equiv \mathscr{L}\left(\phi\right)$) using the path integral formulation and the so-called generating functional $Z\left[J\right]$. This enables us to compute the Green's function starting from the general expression

$$Z\left[J\right] = \int \mathfrak{D}\left[\phi\left(x\right)\right]\ e^{\,i\,\left(\ S_{\mathscr{L}}[\phi]\ +\ S^{(int)}[J,\phi]\ \right)} \tag{3.2.1}$$

and making derivatives. Here $S_{\mathscr{L}}\left[\phi\right]$ is the field action and $S^{(int)}\left[J,\phi\right]$ is the coupling with an external source J. Equation (3.2.1), in this form, expresses,

up to a normalization constant, the transition amplitude between ground states of the theory $Z[J] = N\langle 0|0\rangle$ (cfr. eq. (2.2.9)) in presence of an external source J coupling to the field ϕ. The external source is responsable for new particles to be created, scattered and then annihilated by the field.
We want to use this method for the gravitational field, starting from the Plebanski formulation of the BF-action; we do not include any coupling with an external source J, so any possibility for new particles to be created or annihilated is taken into account only by the self-interaction (if present) of the field itself.
We start from the unconstrained BF-theory in any dimension D and try to implement the discretization case by case: in $1+1$ dimensions the BF-theory in just a topological theory, the action being written using only the curvature $F = D\omega$; in $2+1$ dimensions, on the contrary, this is exactly General Relativity with first order formalism. In $3+1$ dimensions, finally, we have to implement the simplicity condition for the B field in order to obtain the Einstein equations.
The generating functional for the Plebanski action is given by:

$$Z(M) = \int \mathfrak{D}\omega\ \mathfrak{D}B\ e^{\,i\int_M tr[B\wedge F]} \tag{3.2.2}$$

where we have put emphasis on the fact that Z is strictly dependent on the manifold M, because gravitation is nothing but a metric theory constructed out of the differential structure assigned on the space-time M.

One can formally integrate out the B field in (3.2.2) obtaining a sort of δ measure on the space of connections ω:

$$Z(M) = \int \mathfrak{D}\omega\ \mathfrak{D}B\ e^{\,i\int_M tr[B\wedge F]} = \int \mathfrak{D}\omega\ \delta(F) \tag{3.2.3}$$

where the δ measure should be interpreted as a sort of "volume" of the space of connections, but, of course, this does not mean anything until we specify the measure $\mathfrak{D}$ appearing in the path integral. The measure itself is indeed an ill-defined quantity, because quantum field theory does not provide any information about the convergence of the path integral itself, this being only a formal expression. Now the problem is that the integral in equation (3.2.3) is divergent, so we have to take a regularization in order this integral to make sense. The regularization we choose emerges as a discretization of the action using triangulations on M. Then we introduce the notion of skeleton dual

with respect to an assigned triangulation and we work with connections on these dual skeltons instead of connections on the discretized space-time M. Given a triangulation $\mathcal{T}$ on a manifold M, the dual 2-skeleton $\mathcal{S}_{\mathcal{T}}$ is defined by making the following assignments: if D is the dimension of the space-time manifold, to each n-simplex is assigned a $D-n$ complex (not necessarily a simplex). For instance, in the 2-dimensional case we have faces, edges and vertices, thus the assignment is obtained in the following way: to each face is assigned a vertex, to each edge is assigned an edge and to each vertex is assigned a face. More interesting is the 3-dimensional case, where we have tetrahedra, faces, edges and vertices: to each tetrahedron we assign a vertex in centre of the complex, to each face corresponds an edge, to each edge corresponds a polygonal face and to each vertex corresponds a hypervolume sorrounding the vertex. The 4-dimensional case works in the same way.

To each one of these elements one refers as to a dual vertex, a dual edge and a dual face. Notice that the polygonal faces we assign to each edge in the original triangulation can have any number of edges themselves. For example, if we use triangulations on the original manifold M, it is not required that the dual faces sorrounding the vertices are triangles themselves. It can look like *Fig.* 3.1.

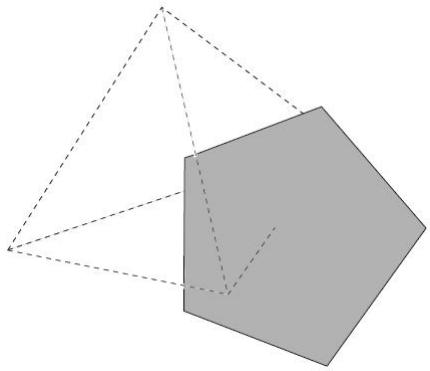

Figure 3.1: *This is a possible polygonal face dual to an edge.*

Each edge of the polygonal face f can be labelled, once we choose an orientation, as[2] $e_1^{*\,(f)} \dots e_N^{*\,(f)}$, and each vertex of such a face as $v_1^{*\,(f)} \dots v_N^{*\,(f)}$.

[2] We use the * symbol to put emphasis on the fact that these edges and vertices refer to the dual skeleton.

Thus we get, for *Fig.* 3.1, the corresponding *Fig.* 3.2:

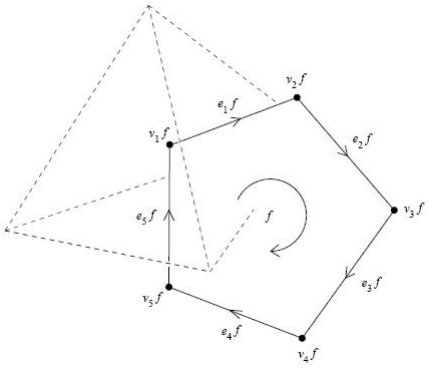

Figure 3.2: *This is the labelling of the polygonal face, provided an orientation is chosen.*

The 2-forms in equation (3.2.2) must be evaluated ("integrated") on the edge of the complex $\mathcal{T}$. We want the curvature F to be *local*, in the sense that it changes if calculated in different points of the space-time. At this point we try to make explicitly the calculations needed to discretize the BF-theories, according to the chosen number of dimensions for the space-time.

The 2-dimensional model. In the case of a two-dimensional space-time, the BF-action simply reduces to

$$S[\,\omega\,] = \int_M tr\,(\,B\,F\,) \tag{3.2.4}$$

B playing the role of a function (Lagrange multiplier). As a consequence, this theory will appear as a purely topological one, with no additional dynamical degrees of freedom.
In this context we say that a connection is assigned to each edge of the dual skeleton if we associate an element $g_{e^*_K}$ to each dual edge e^*_K. This assignement may come from integrating the connection one-form ω on the dual edge e^* and then taking the exponential as in equation (1.5.14), provided a mathematical definition of integration on dual edges to exist. The "curvature" of the dual face f is obtained by taking the product of all the group elements assigned to its edges:

$$g_{f^*} = \prod_{e^* \subset \partial f^*} g_{e*} \tag{3.2.5}$$

A connection on the dual skeleton of a triangulation is said to be *flat* if the curvature is the identity element, $g_{f^*} = \hat{1}$, for each face f of the skeleton. This description is a sort of spin network formulation on the set of dual skeletons, in fact we have vertices, faces and edges labelled with group elements; thus spin networks can also be regarded as abstract objects associated to some representations of a gauge group G, and not only as diagrams embedded in the space-time manifold M. In this spirit the states of the gravitational field will appear as purely algebraic elements not required to emerge from any discretization of the space-time, and the informations about a triangulated manifold is translated into the language of combinatorics and group theory.

The procedure we introduce in order F to be "locally" discretized is the following: ω is a Lie-algebra valued one form, so it must be smeared on an edge, and we choose it to be a dual edge e^*

$$\int_{e^*} \omega \ = g_{e^*} \in G \tag{3.2.6}$$

Then we take the element defined by

$$g_{f^*} = \prod_{e^* \subset \partial f^*} g_{e*} = \prod_{e^* \subset \partial f^*} e^{\int_{e^*} \omega} \ . \tag{3.2.7}$$

that is, the holonomy on a face in the dual 2-skeleton $\mathcal{S}_\mathcal{T}$ represents the local curvature of the bundle. Because in the 2-dimensional case the dual face f^* of the dual 2-skeleton $\mathcal{S}_\mathcal{T}$ corresponds to a vertex v in the original simplex, we introduce the notation

$$F_v = \prod_{e^* \subset \partial f^*} e^{\int_{e^*} \omega} \ . \tag{3.2.8}$$

As a consequence, the BF-action becomes, in the discretized case

$$S\,[\,\omega\,] = \int_M tr\,(\,B\,F\,) = \sum_{v \in \mathcal{T}} tr\,(\,B\,F_v\,) \tag{3.2.9}$$

and the generating functional reads

$$Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}\,(M) = \int \mathfrak{D}\omega\ \mathfrak{D}B\ e^{\ i\ \sum\limits_{v \in \mathcal{T}} tr(B\,F_v)} \tag{3.2.10}$$

At this stage, this description allows us to substitute the integration over $\mathfrak{D}\omega$ in equation (3.2.3) with an integration on the group variables $dg_{e^*_K}$, and

then use group theory techniques to evaluate explicitly the calculation. The exponential appearing in the integral above can be integrated out giving raise to a δ functions whose arguments are the product of elements g_{e^*} on the group, as shown in equation (3.2.7). Thus the generating functional writes

$$Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}(M) = \int \prod_{e^* \in \mathcal{E}} dg_{e^*} \prod_{f^* \in \mathcal{F}} \delta\left(g_{e_1^*(f)} \dots g_{e_N^*(f)}\right) \tag{3.2.11}$$

where $\mathcal{E}$ and $\mathcal{F}$ are respectively the set of all the dual edges and dual faces of the dual skeleton $\mathcal{S}_\mathcal{T}$. Remark that the sum could equivalently run over the vertices of the original complex $\mathcal{T}$, because, in the two-dimensional case, faces f^* in the dual skeleton correspond to vertices v in the original complex $\mathcal{T}$. We want to stress (by using the subscripts) the fact that this generating functional strictly depend on the triangulation $(\mathcal{T}, \mathcal{S}_\mathcal{T})$ chosen on the manifold M through the choice of the edges and elements $g_{e_K^*}$ attached to each edge.

The measure dg appearing in the integral is the Haar measure on the gauge group G, which now we recall to be the group SU (2), in order to reduce this two-dimensional BF-theory to General Relativity. In fact we can take into account the full invariance under the gauge group SO(2) of pure rotations in two dimensions and under the group of diffeomorphisms by taking the connection one-form ω, which is SO(2)-valued, and extending it to a SU(2)-valued one. At this point we perform the calculations by using pure group theory techniques; what we need is the decomposition of the delta function on the group in terms of irreducible representations:

$$\delta(g) = \sum_{\rho \in Irr\ rep(G)} \dim(\rho)\ \ tr[\ \rho(g)\ ] \quad . \tag{3.2.12}$$

We introduce this formula into equation (3.2.11) to obtain

$$Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}(M) = \int \prod_{e^* \in \mathcal{E}} dg_{e^*} \prod_{f^* \in \mathcal{F}} \sum_{\rho \in Irr\ rep(G)} \dim(\rho_{f^*})\ \ tr[\ \rho_{f^*}(g_{f^*})\ ] \tag{3.2.13}$$

and intend this expression to be a discretized version of (3.2.2), because the measure $\mathfrak{D}\omega$ has been substituted by a Haar measure dg on the gauge group. In order to perform the explicit calculation of (3.2.13) we notice that, in the two-dimensional case, each face is closed by two edges, so the argument in $[\ \rho_{f^*}(g_{f^*})\ ]$ is the product of two elements of the group, i.e. we should

write $[\ \rho_{f^*}(g_1, g_2)\ ]$. By definition of representation, the representation of a product is the product of the representations, so we have $[\ \rho_{f^*}(g_1, g_2)\ ] = [\ \rho_{f^*}(g_1)\ \rho_{f^*}(g_1)\ ]$. Moreover, each dual edgee^*, in the two-dimensional case, is shared by two dual faces f^*; as a consequence, the product of the traces over the dual facese splits into the product of two different representations, one for each vertex in the dual skeleton, $v^* \in \mathcal{V}$. In the case of $G = \mathrm{SU}(2)$, the above integral (3.2.13) may be written as

$$\begin{aligned} Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}(M) &= \prod_{\substack{f^*\in\mathcal{F}\\ v^*\in\mathcal{V}}} \sum_{n,m\in\mathbb{Z}} \dim\left(\rho_{f^*}^{(n,m)}\right) \prod_{e^*\in\mathcal{E}} \int_{SU(2)} dg_1\ dg_2\ {D^{(j)}_{m,n}}^{f*_1}(g_1) \times \\ &\times\ {D^{(j)}_{n,l}}^{f*_1}(g_2)\ {D^{(j')}_{r,s}}^{f*_2}(g_1)\ {D^{(j')}_{s,k}}^{f*_2}(g_1) \quad . \end{aligned} \tag{3.2.14}$$

Now we make use of the orthogonality condition for the product of two matrix elements of the representations of SU (2)

$$\int_{SU(2)} dg\ D^{j}_{m,n}(g)\ \left(D^{j'}_{m',n'}(g)\right)^* = \frac{1}{\dim(\rho)}\ \delta^{jj'}\,\delta_{mm'}\,\delta_{nn'} \tag{3.2.15}$$

so that we have

$$Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}(M) = \prod_{\substack{f^*\in\mathcal{F}\\ v^*\in\mathcal{V}}} \sum_{j\in\frac{\mathbb{N}}{2}} \dim\left(\rho_{f^*}^{(j)}\right) \prod_{e^*\in\mathcal{E}} \frac{1}{\dim(\ \rho^{(j)}\ )} \tag{3.2.16}$$

and, as a consequence:

$$Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}(M) = \sum_{j\in\frac{\mathbb{N}}{2}} \dim\left(\rho^{(j)}\right)^{|\mathcal{V}|-|\mathcal{E}|+|\mathcal{F}|} \tag{3.2.17}$$

If we choose the manifold M to be connected and use discrete regularization obtained by means of polyhedra on M, the quantity $|\mathcal{V}|-|\mathcal{E}|+|\mathcal{F}| = \chi(M)$ is a topological invariant depending only on the manifold M, namely the Euler characteristic. Thus equation (3.2.13) becomes, in its final form:

$$Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}(M) = \sum_{\rho\in Irr\ rep(G)} \dim(\rho)^{\chi(M)} \quad . \tag{3.2.18}$$

The Euler characteristic is strictly connected to the genus of a manifold, i.e. the maximum number of closed simple curves one can draw on the manifold without taking the resulting manifold into disconnected submanifolds. If g is the genus of a given manifold, then its Euler characteristic is $\chi(M) = 2 - 2g$.

When $\chi(M) < 0$ the sum on the right hand side of equation (3.2.18) converges for any compact Lie group G, and in this case the generating functional is well-defined and, above all, independent of the triangulation chosen, because it depends only on a topological property of the manifold M. On the contrary, for any manifold of genus $g > 1$, the generating functional is typically not convergent, the result not depending on the chosen discretization. We regard this partition function as a transition amplitude, fixed a particular geometry $(\mathcal{T}, \mathcal{S}_{\mathcal{T}})$; thus we should regard it as the single term appearing in right hand side in the expression (2.2.11) to be summed over in order to obtain the total generating functional of the theory, the sum running over all the possible triangulations (i.e. geometries) admitted on the space-time manifold.

The 3-dimensional model. In the case of a three-dimensional space-time, the BF-action writes

$$S[\ \omega\ ,\ B\ ] = \int_M tr(\ B \wedge F\ )\ . \tag{3.2.19}$$

The B-field appearing in the integral above must be a $3-2=1$–form valued in the Lie-algebra of the gauge group G, which now is the group $\mathrm{SO}(3)$ of rotations in three dimensions. We can choose B to be the tetrad tensor field e, and in this case the action (3.2.19) represents exactly the General Relativity, with no other constraints to be imposed, because the B-field is, by construction, a one-form. This one-form can be integrated over an edge of the complex $\mathcal{T}$, in order to obtain an element of the group G

$$B_e = \int_e B\ . \tag{3.2.20}$$

In the same way we can integrate the connection one form ω on a dual edge e^*. This is because, by taking the holonomy

$$F_e = \prod_{e^* \subset \partial f^*} e^{\int_{e^*} \omega} \tag{3.2.21}$$

we can construct the local curvature on each edge e of the original complex (recall that in the 3-dimensional case to each face f^* in the dual skeleton $\mathcal{S}_{\mathcal{T}}$ corresponds an edge in the original complex $\mathcal{T}$).
Now the discretized action can be written in the same way of equation (3.2.9),

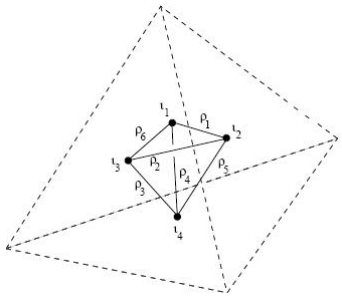

Figure 3.3: 3-*dimensional tetrahedron containing its own dual skeleton.*

with the difference that now we are going to sum over the edges of the original complex $\mathcal{T}$

$$S\left[\ \omega\ ,\ B\ \right] = \int_M tr\left(\ B \wedge F\ \right) = \sum_{e \in \mathcal{T}} tr\left(\ B_e\, F_e\ \right) \tag{3.2.22}$$

The sum could equivalently run over the dual faces $f^* \in \mathcal{S}_\mathcal{T}$, because in the three-dimensional case edges of the original complex correspond to faces in the dual skeleton. The generating functional now writes:

$$Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}\left(M\right) = \int \mathfrak{D}\omega\ \mathfrak{D}B\ e^{\ i\ \sum\limits_{e\in\mathcal{T}} tr(B_e\, F_e)} \ . \tag{3.2.23}$$

The key point now is to reduce the integration over the variables ω and B to an integration on the group variables $dg_{e^*_K}$, and integrate out the B_e fields. This procedure furnishes an expression exactly identical to (3.2.11). That formula can be reduced, as we have shown in the previous, to equation (3.2.13)

$$Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}\left(M\right) = \int \prod_{e^* \in \mathcal{E}} dg_{e^*} \prod_{f^* \in \mathcal{F}} \sum_{\rho \in Irr\ \ rep(G)} \dim\left(\rho_{f^*}\right)\ \ tr\left[\ \rho_{f^*}\left(g_{f^*}\right)\ \right]\ . \tag{3.2.24}$$

Because now the gauge group is $G = \mathrm{SO}\left(3\right)$, we can perform the calculation using the explicit form of the representation for this group. The generating functional thus becomes:

$$\begin{aligned} Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}\left(M\right) \ &= \ \int \prod_{e^* \in \mathcal{E}} dg_{e^*} \prod_{f^* \in \mathcal{F}} \sum_{\rho \in Irr\ \ rep(G)} \left(2j_{f^*}+1\right) \times \\ &\times \ \ tr\left[\ \rho_{f^*}\left(g_{f_1^*}\right)\ \rho_{f^*}\left(g_{f_2^*}\right)\ \rho_{f^*}\left(g_{f_3^*}\right)\ \right] \end{aligned} \tag{3.2.25}$$

where we have used the fact that now a face, in the three-dimensional case, is closed by three edges, so the argument of the representation $[\ \rho_{f^*}\,(g_{f^*})\ ]$ is the product of three elements of the group, and so the representation can be split into the product of three representations, one for each element. We can decompose the representations on an orthonormal basis $|\,j\,m\,\rangle$ in the vector space of dimensions $2j+1$, on which the j^{th} representation acts. Thus we get

$$\begin{aligned} Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}\,(M) \;&=\; \int \prod_{e^*\in\mathcal{E}} dg_{e^*} \prod_{f^*\in\mathcal{F}} \sum_{\rho\in Irr\ \ rep(G)} (2j_{f^*}+1)\ \times \\ &\times\ \sum_{\{m,n\}} D^{(j)}_{m,n}\,(g_1)\ D^{(j')}_{m',n'}\,(g_2)\ D^{(j'')}_{m'',n''}\,(g_3)\times \\ &\times\ \ tr\,[\ |\,j\,m\,\rangle\langle\,j\,n\,|\,j'\,m'\,\rangle\langle\,j'\,n'\,|\,j''\,m''\,\rangle\langle\,j''\,n''\,|\ ] \end{aligned} \tag{3.2.26}$$

Taking into account the orthonormality conditions for the basis $|\,j\,m\,\rangle$, and the fact that $tr\,[\ |\,\alpha\,\rangle\langle\,\beta\,|\ ] = \langle\,\beta\,|\,\alpha\,\rangle$, we get

$$\begin{aligned} Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}\,(M) \;&=\; \int \prod_{e^*\in\mathcal{E}} dg_{e^*} \prod_{f^*\in\mathcal{F}} \sum_{\rho\in Irr\ \ rep(G)} (2j_{f^*}+1)\ \times \\ &\times\ \sum_{\{m,n\}} D^{(j)}_{m,n}\,(g_1)\,D^{(j)}_{n,n'}\,(g_2)\ D^{(j)}_{n',m}\,(g_3) \end{aligned} \tag{3.2.27}$$

Now we have to take into account the fact that each edge is shared by three dual faces, so the product on $f^*\in\mathcal{F}$ splits into three different traces, one for each face; thus we obtain:

$$\begin{aligned} Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}\,(M) \;&=\; \prod_{f^*\in\mathcal{F}} \sum_{\rho\in Irr\ \ rep(G)} (2j_{f^*}+1) \int \prod_{e^*\in\mathcal{E}} dg_{e^*}\times \\ &\times\ \sum_{\{m,n,n'\}} D^{(j_1)\,f_1^*}_{m,n}\,(g_1)\,D^{(j_1)\,f_1^*}_{n,n'}\,(g_2)\ D^{(j_1)\ f_1^*}_{n',m}\,(g_3)\times \\ &\times\ \sum_{\{r,s,s'\}} D^{(j_2)f_2^*}_{r,s}\,(g_1)\,D^{(j_2)f_2^*}_{s,s'}\,(g_2)\ D^{(j_2)f_2^*}_{s',r}\,(g_3)\times \\ &\times\ \sum_{\{k,l,l'\}} D^{(j_3)f_3^*}_{k,l}\,(g_1)\,D^{(j_3)f_3^*}_{l,l'}\,(g_2)\ D^{(j_3)f_3^*}_{l',k}\,(g_3)\quad . \end{aligned} \tag{3.2.28}$$

We can perform the integration over the group variables dg_1, dg_2 and dg_3. At this stage we make use of the expression furnishing the integral of the

product of three matrix elements of the representations on the group SO (3) in terms of the Wigner $3j$-symbols:

$$\int_G dh\ D^{j_1}_{n_1,k_1}(h)\ D^{j_2}_{n_2,k_2}(h)\ D^{j_3}_{n_3,k_3}(h) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \tag{3.2.29}$$

where (see also the Appendix A.7)

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2+n_3}}{\sqrt{2j_3+1}}\ \langle j_1\, j_2;\ n_1\, n_2 \,|\, j_1\, j_2;\ j_3\ -n_3 \rangle\ . \tag{3.2.30}$$

We thus get the product of two $3j$-symbols for each dual edge e^*, i.e. for each triangular face of the original complex $\mathcal{T}$; in particular we have:

$$Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}(M) = \prod_{f^*\in\mathcal{F}} \sum_{\rho\in Irr\ rep(G)} (2j_{f^*}+1) \prod_{e^*\in\mathcal{E}} \{3j\} \tag{3.2.31}$$

Each triangular face of the complex $\mathcal{T}$ (i.e. each dual edge $e^* \in \mathcal{S}_\mathcal{T}$) may be glued to several tetrahedra; the indices referring to the same tetrahedron are contracted one other, so we have a full contraction of the $3j$-symbols referring to same tetrahedra, i.e. to the same vertex $v^* \in \mathcal{S}_\mathcal{T}$. In particular, if we introduce the Racah-Wigner $6j$-symbols:

$$\begin{aligned} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} &= \sum_{-j_i\le m_i\le j_i} (-1)^{j_4+j_5+j_6+m_4+m_5+m_6} \times \\ &\times \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_5 & j_6 & j_1 \\ m_5 & -m_6 & m_1 \end{pmatrix} \times \\ &\times \begin{pmatrix} j_6 & j_4 & j_2 \\ m_6 & -m_4 & m_2 \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_3 \\ m_4 & -m_5 & m_3 \end{pmatrix} \end{aligned} \tag{3.2.32}$$

we can write the equation (3.2.31) in a more compact form:

$$Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}(M) = \prod_{f^*\in\mathcal{F}} \sum_{\rho\in Irr\ rep(G)} (2j_{f^*}+1) \prod_{v^*\in\mathcal{V}} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}_{v^*} \tag{3.2.33}$$

This is the final form we get for the generating functional in the 3-dimensional case, $\mathcal{V}$ being the set of all vertices in $\mathcal{S}_\mathcal{T}$.

The 4-dimensional model. The BF-action in 4-dimensions is

$$S[\ \omega\ ,\ B\ ] = \int_M tr\ (\ B \wedge F\ )\ . \tag{3.2.34}$$

where now the B-field must be a 2-form. In order this action to give raise to the equation of General Relativity, one has to require further condition on the B-field to be satisfied. In particular, B must be expressed as the wedge product of two one-forms. In the following section we are going to introduce the Barrett-Crane model, that is a model implementing the condition for B at the level of the representations of the gauge group $G = \mathrm{SO}(4)$; in this paragraph, on the contrary, we only derive the expression of the generating functional for a 4-dimensional BF-theory without imposing any constraints. The discretization of the BF-action works similar to the other cases; recall that now a 4-simplex is made out of five tetrahedra, ten triangles, then edges and five vertices. Here both B and F are 2-forms, so they have to be integrated on a face, and we choose a face $f \in \mathcal{T}$. In this way we obtain

$$B_f = \int_f B \tag{3.2.35}$$

and

$$F_f = \prod_{e^* \subset \partial f^*} e^{\int_{e^*} \omega} \tag{3.2.36}$$

The discretized action is

$$S[\ \omega\ ,\ B\ ] = \int_M tr(\ B \wedge F\ ) = \sum_{f \in \mathcal{T}} tr(\ B_f\, F_f\ ) \tag{3.2.37}$$

where we could also sum over the dual faces $f^* \in \mathcal{S}_\mathcal{T}$, because in the four-dimensional case the dual corrispondence maps faces $f \in \mathcal{T}$ into faces $f^* \in \mathcal{S}_\mathcal{T}$. The generating functional assumes the form

$$Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}(M) = \int \mathfrak{D}\omega\ \mathfrak{D}B\ e^{\ i \sum\limits_{f \in \mathcal{T}} tr\left(B_f\, F_f\right)}\ . \tag{3.2.38}$$

It's easy to convince oneself that also in this case we obtain, by performing the integration of the exponential, an expression containing the product of δ functions, which can be reduced as equation (3.2.13)

$$Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}(M) = \int \prod_{e^* \in \mathcal{E}} dg_{e^*} \prod_{f^* \in \mathcal{F}} \sum_{\rho \in Irr\ \ rep(G)} \dim(\rho_{f^*})\ \ tr[\ \rho_{f^*}(g_{f^*})\ ] \tag{3.2.39}$$

Now again the trace can be decomposed into the product of four different traces, one for each face (because an edge is now shared by four dual faces), each one of such traces containing the product of representations of the gauge

group in each point $g_{e^*} \in G$. The key point is now that we want to regard the group $\mathrm{SO}(4)$ as the product $\mathrm{SU}(2) \times \mathrm{SU}(2)$, so an element $g \in \mathrm{SO}(4)$ can be regarded as a pair (g^+, g^-) $g^+, g^- \in \mathrm{SU}(2)$. Moreover, a representation of $\mathrm{SO}(4)$ can be regarded as a pair (j^+, j^-) of representation of $\mathrm{SU}(2)$.
The calculation proceeds in the same way as in the 3-dimensional case, with the difference that now we have to decompose the sum over the irreducible representations ρ of $\mathrm{SO}(4)$ into the product of two sums over the irreducible representations j^+ and j^- of $\mathrm{SU}(2)$. This procedure can easily understood in terms of decomposition of irreducible representation as stated by the Peter-Weyl theorem on the harmonic analysis on groups. Because the calculation follows the same details as in the three-dimensional case, we obtain an expression quite similar to the (3.2.33), with the difference that now the sum over the irreducible representations has been splitted into two sums, and a dual edge $e^* \in \mathcal{S}_\mathcal{T}$ may be glued to different 4-simplexes. Thus, as a consequence, we have a more complicated contraction of the Wigner $3j$-symbols, giving raise to the so-called $15j$-symbols, whose explicit form is shown in the Appendix A.7. The expression for the generating functional we obtain, in the four-dimensional case, is the following:

$$Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}(M) = \prod_{f^* \in \mathcal{F}} \sum_{\substack{\rho^+,\rho^- \in \\ Irr\ rep(G)}} \left(2j^+_{f^*}+1\right)\left(2j^-_{f^*}+1\right) \prod_{v^* \in \mathcal{V}} \left\{15j^+\right\}_{v^*} \left\{15j^-\right\}_{v^*} \tag{3.2.40}$$

We remark that at this point no constraints have been imposed on the BF-action (even in its discretized form), in order to reduce this formulation to General Relativity. In the following section we show how to introduce these constraints at the level of the representations (j^+, j^-) of $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$.

These calculations furnishes the generating functional for the BF theories, given a particular choice of the triangulation $(\mathcal{T}, \mathcal{S}_\mathcal{T})$ of the space-time. These theories, according to the dimension of the space-time manifold, may or may not reduce to General Relativity, depending on the constraints imposed on the B-field. At this level, the fundamental point of view is that we do not want to regard the gravitational interaction as a theory *on* the space-time, but as a theory *of* the space-time; so, when we choose a particular discretization of the carrier manifold M, we are evaluating transition amplitudes for a given geometry, i.e. a given "state" of the gravitational field. In this spirit the

generating functional $Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}(M)$ can be regarded as a particular Feynamn diagram which takes into account the contribution of the state represented by the triangulation $(\mathcal{T},\mathcal{S}_\mathcal{T})$ (i.e. the "geometry") to an overall partition function whose propagators and Feynman diagrams amplitudes are given by expressions like $Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}(M)$, for each possible triangulation $(\mathcal{T},\mathcal{S}_\mathcal{T})$.
We will see, in chapter 4, a possible theoretical formulation where, starting from a suitable action, we generate all such Feynman diagrams. It has been shown in [5] that such an action term does exist for 3-dimensional discrete formulations of gravity, which reproduces, as Feynman rules, the amplitudes (3.2.11). Such discrete path integral models, whose Feynman amplitudes are given by the partition functions $Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}(M)$, are called "Group field theories".

3.3 The Barrett-Crane model

In this section we introduce the Barrett-Crane model for quantum gravity. In the previous chapters we have shown how the fundamental combinatoric elements, useful to implement the quantum field theory approach to the quantum gravity, are the so-called spin foams. These spin networks first emerged as kinematical states of quantum general relativity out of which one can reconstruct a Hilbert space of quantum states; then they were regarded as purely algebraic objects used to construct a discrete path integral formulation of the gravitational field.
The Barrett-Crane model is a spin foam description of $G = \mathrm{SO}(4)$ Plebanski's formulation of General Relativity. Because of the presence of the gauge group $\mathrm{SO}(4)$, we say that this model is an Euclidean formulation of quantum gravity in 4 dimensions, based on a triangulation of the 4-dimensional space-time in terms of 4-simplexes. The most important feature of the Plebanski formulation is that it is a constrained BF-theory in which one tries to implement the simplicity condition for the B field to be expressed as a wedge product of one-forms $B = e \wedge e$ [3], in order to reduce the constrained

[3] Some authors refer to this fact by saying B is a simple *bivector*.

BF-theory to General Relativity; in the classical description this is easily achieved by introducing a Lagrange multiplier field into the BF-actions so that the equations of motion state the B field to be simple. The solutions of such equations can be split into two different sectors (cfr eq. (3.1.3) and (3.1.4)) and classically one can choose which one sector include without interferences between the solutions. At the quantum level this is not possible, because the path integral quantization shares a sum over all possible configurations of the B field, gluing together the two different sectors. As a consequence, the simplicity condition for the B field must be introduced in a different way; this is exactly what the Barrett-Crane model tries to pursue.

We start from a discretization of the 4-dimensional space-time in terms of geometric 4-simplexes, which will play the role of the spin foams. A geometric 4-simplex $\wp$ is the convex hull of 5 points in $\mathbb{R}^4$ [4], and it is completely characterized by a set of 10 bivectors $b_i \in \mathbb{R}^4 \wedge \mathbb{R}^4$, each bivector corresponding to a triangle in the 4-simplex. For the characterization to be unique the following conditions must hold:

- Each bivector is simple, i.e. it is a wedge product of two one-forms (or two one-vectors).
- Each bivector is non-degenerate.
- The sign of a bivector agrees with the orientation of the triangle; if the orientation is reversed then the bivector changes in sign.
- If two triangles share an edge, the sum of their corresponding bivectors is required to be a simple bivector.
- The oriented sum (i.e. taking into account the signs of the bivectors) of the bivectors corresponding to the faces of a tetrahedron must be zero.
- The bivectors corresponding to triangles meeting at a given vertex of the tetrahedron must satisfy the condition $tr\ (\ b_1\ [b_2, b_3]\ ) \geq 0$

If these conditions hold then the 4-simplex $\wp$ is uniquely characterized by the assignement of 10 bivectors. Now the crucial point is that we want to

[4] This set of points is required to be non-degenerate, i.e., if D is the dimension of the space-time manifold, no $D-1$-hyperplanes contain all the $D+1$ points.

relate each bivector $b_i \in \mathbb{R}^4 \wedge \mathbb{R}^4$ to an element in the Lie-algebra $\mathfrak{g}$ of the gauge group G. In order to achieve this step we need an isomorphism which translate the bivectors

$$\theta : \mathbb{R}^4 \wedge \mathbb{R}^4 \longrightarrow \mathfrak{so}\,(4) \ . \tag{3.3.1}$$

Recalling the notations used in § 3.1, such a map is obtained by taking

$$X_i^{IJ} = \frac{1}{2}\ \epsilon_{KL}{}^{IJ}\ b_i^{KL} \tag{3.3.2}$$

where X^{IJ} is an element in the Lie-algebra $\mathfrak{so}\,(4)$ and b_i^{KL} is the bivector associated to the i^{th} triangle in the 4-simplex $\wp$. Here $\epsilon_{KL}{}^{IJ}$ is a linear antisymmetric tensor which takes into account the antisymmetry of the generators of $\mathfrak{so}\,(4)$.

At this point each bivector can be thought as an element in the Lie-algebra $\mathfrak{so}\,(4)$, and thus as emerging out of the B-field present in the BF-action, for example through the procedure of assignement of an holonomy, as we did in § 1.5.1. In this way, if Δ is a graph of a spin network, i.e. a triangle in the 4-simplex $\wp$, then the assignement of a bivector B_Δ by means of the holonomy is

$$B_\Delta = \int_\Delta B \ . \tag{3.3.3}$$

This is indeed a Lie-algebra element associated to the triangle $\Delta \in \wp$, after performing the corrispondence exhibited in (3.3.1). In particular the map (3.3.2) thus states that

$$X_\Delta^{IJ} = \frac{1}{2}\ \epsilon_{KL}{}^{IJ}\ B_\Delta^{KL} \tag{3.3.4}$$

and, if the simplicity condition in the first sector (equation (3.1.3)) is $B^{KL} = \pm e^K \wedge e^L$, then

$$X_\Delta^{IJ} = \pm\ \frac{1}{2}\ \epsilon_{KL}{}^{IJ}\ e^K \wedge e^L \ . \tag{3.3.5}$$

The fundamental step now is that we want to regard the Lie-algebra $\mathfrak{so}\,(4)$ as the direct sum of $\mathfrak{su}\,(2)$ algebras

$$\mathfrak{so}\,(4) = \mathfrak{su}\,(2) \oplus \mathfrak{su}\,(2) \tag{3.3.6}$$

so that the labelling of each triangle with an element $X^{IJ} \in \mathfrak{so}\,(4)$ is the same as the labelling of such a triangle with a couple of $\mathfrak{su}\,(2)$ elements, that

is to say, by a pair (j^+, j^-) of spins. With the help of this decomposition, the constraint on the B field to be simple, which is the fundamental point that brings a BF-theory into the Einstein theory for the gravitational field, can be imposed as a constraint on the Lie-algebra element $X^{IJ} \in \mathfrak{so}(4)$, and, as a consequence, on the choice of the spin pair (j^+, j^-).

The Lie-algebra element X^{IJ}_Δ associated to each triangle can be decomposed, by means of the splitting (3.3.6), into a sum of two components, namely X^{IJ}_+ and X^{IJ}_- taking values in $\mathfrak{su}(2)$:

$$\begin{array}{rcccc} \mathfrak{so}(4) & = & \mathfrak{su}(2) & \oplus & \mathfrak{su}(2) \\ & & \downarrow & & \downarrow \\ X^{IJ}_\Delta & = & X^{IJ}_+ & + & X^{IJ}_- \\ & & \downarrow & & \downarrow \\ & & (\ j^+ & , & j^-\ ) \end{array}$$

Here we furnish the fundamental result which enables us to translate the simplicity condition for a bivector to be simple into a condition on the representation of the gauge group used in (3.3.1).

Theorem. *A bivector $b^{IJ} \in \mathbb{R}^4$ is a simple bivector if and only if there exists a vector n^I such that $b^{IJ} n_J = 0$.*

Proof. Let us first prove the statement b^{IJ} simple $\Rightarrow \exists\, n^I : b^{IJ} n_J = 0$. If the bivector b^{IJ} is simple then it defines a hyperplane of dimension 2 embedded into $\mathbb{R}^4$; in this case the assertion is proven by taking as n^I any vector orthogonal to that hyperplane.

On the contrary, let us prove the statement $b^{IJ} n_J = 0 \Rightarrow b^{IJ}$ is simple. Assume b^{IJ} not to be simple; in this case there must exist a set of 4 vectors $\{u_1, u_2, v_1, v_2\}$, spanning the entire space $\mathbb{R}^4$, such that b^{IJ} can be expressed in the form

$$b^{IJ} = u_1^I \wedge v_1^J + u_2^I \wedge v_2^J \ . \tag{3.3.7}$$

Because the set of vectors $\{u_1, u_2, v_1, v_2\}$ is a basis of $\mathbb{R}^4$, then a generic vector, say n^I, can be constructed out of a linear combination of them. In this case the condition $b^{IJ} n_J = 0$ imposes a linear relation between the four vectors $\{u_1, u_2, v_1, v_2\}$, and thus they cannot be linearly independent, i.e. a

basis of $\mathbb{R}^4$. This contradicts the assumption that b^{IJ} is not simple.

□

The condition for the X^{IJ} field to be simple translates into a condition for the norms of X^{IJ}_+ and X^{IJ}_- to be equal. In fact, if we express the discretized B_Δ in terms of X^{IJ}_Δ, by inverting equation (3.3.4), we get

$$\left(\epsilon_{KL}{}^{IJ}\ X^{KL}\right)\ n_J = 0 \tag{3.3.8}$$

whose solution, in terms of X^{IJ}_+ and X^{IJ}_-, writes

$$X^{IJ}_+ = X^{IJ}_- \Longrightarrow j^+ = j^- \ . \tag{3.3.9}$$

Thus, to impose this constraint at the quantum level, it is natural to restrict to those configurations of labellings (j^+, j^-) of triangles for which these spins are equal. Now we can substitute back the condition we have found in terms of labelling of the representation of the gauge group, into the expression (3.2.40) of the generating functional for 4-dimensional BF-theories. In this way, we can reduce the BF-theory to the first sector, (3.1.3), of the Einstein equation for General Relativity. The partition function thus gets the form:

$$Z_{[\mathcal{T},\mathcal{S}_\mathcal{T}]}(M) = \prod_{f^*\in\mathcal{F}} \sum_{j\in Irr\ rep(G)} (2j_{f^*}+1)\ (2j_{f^*}+1) \prod_{v^*\in\mathcal{V}} \{15j\}^2 \ . \tag{3.3.10}$$

In conclusion, the Barrett-Crane model consists of 3 main steps: first, a discretization of the carrier manifold M has been taken in terms of 4-simplexes $\wp$, each of them characterized by a set of 10 bivectors $b_i \in \mathbb{R}^4 \wedge \mathbb{R}^4$; then we introduce a corrispondence θ (equation (3.3.1)) between bivectors and Lie-algebra elements. The last step is the quantization of the constraint equation for the B field, obtained by means of techniques of group representation theory; the simplicity constraint is imposed at the level of the representations of the gauge group (in this case $G = \mathrm{SO}(4)$) used to label each 4-simplex $\wp$. Indeed the Barrett-Crane model shows a problem, because this formulation can quantize only one of the two sectors of classical solution of the constraint equation for B, namely equation (3.1.3). The complete classical description of the gravitational field achieved in terms of the Palatini-Holst action states B to be a linear combination $B = a_1\ e \wedge e + a_2\ *(e \wedge e)$, with a_1 and a_2 not simultaneously zero. This easily leads to the description in terms of the Ashtekar variables using the Immirzi parameter, which is the starting point for the Hamiltonian formulation of the gravitational field in terms of

canonical Poisson brackets between the conjugate variables A_a^i and E_j^b (cfr. pag. 29) . The Barrett-Crane model cannot take into account the presence of an Immirzi parameter in the form of a_2/a_1 (or viceversa, a_1 and a_2 being the coefficients of the linear combination in (3.1.7)), because the simplicity condition at the level of the choice of the representation (j^+, j^-) of the gauge group only implements $B = e \wedge e$; we refer to this fact by saying that the Barrett-Crane model can describe only one sector of the constraint equations for B, and thus this model is only a topological description of the gravitational field, because it drops out the further degrees of freedom included in the choice of an Immirzi parameter different from $\gamma = \infty$ in (3.1.7).
The attempt to achieve a complete quantization procedure which takes into account a generic Immirzi parameter γ is pursued with the so-called EPRL/FK model, which we are going to describe in the next section.

3.4 The EPRL/FK model

In this section the EPRL/FK[5] quantization procedure for a BF-action is described. The main difference between this model and the Barrett-Crane one is that now we try to introduce the Immirzi parameter into the quantum version of the constraint equation for the B field, in order to obtain a quantization procedure able to describe the full Palatini-Holst action, without choosing only one sector of solution for the simplicity condition. The key point is that the introduction of the Immirzi parameter is achieved at the level of the map (3.3.1) which assignes to each bivector an element $\in \mathfrak{so}(4)$. In fact a linear combination of the two sectors of solutions (3.1.3) and (3.1.4) states that B must be of the form:

$$B^{IJ} = e^I \wedge e^J + \frac{1}{\gamma}\, \epsilon_{KL}{}^{IJ}\, e^K \wedge e^L \ . \tag{3.4.1}$$

This expression suggest us to modify the map θ in eq. (3.3.2) into

$$X_i^{IJ} = \frac{1}{2}\, \epsilon_{KL}{}^{IJ}\, b_i^{KL} + \frac{1}{\gamma}\, b_i^{IJ} \tag{3.4.2}$$

[5] J. Engle, R. Pereira, C. Rovelli, R.E. Livine, L. Freidel, K. Krasnov.

where we can recognize that, if we let γ approach to infinity into this formula, we obtain the condition for B to be simple only in the first sector of solutions (3.1.3), leading to the Barrett-Crane quantization which does not take into account the presence of the Immirzi parameter into the Palatini-Holst action.

We use the theorem of simplicity for the bivectors b_i associated to each triangle in a 4-simplex $\wp$, i.e. inverting equation (3.4.2). Thus we get:

$$b_i^{IJ} = \frac{\gamma}{1-\gamma^2} \left(X_i^{IJ} - \gamma \, \epsilon_{KL}{}^{IJ} \, X_i^{KL} \right) \tag{3.4.3}$$

and the simplicity condition for b_i^{IJ} is translated into the requirement for a vector n^I to exist such that

$$b^{IJ} n_J = 0 \Longleftrightarrow \left(X^{IJ} - \gamma \, \epsilon_{KL}{}^{IJ} \, X^{KL} \right) n_J = 0 \; . \tag{3.4.4}$$

The solution of this equation provides the constraints to be imposed at the level of the representation of the Lie-algebra element X_i^{IJ}, as a consequence of the simplicity condition imposed on the B field. In this context the Barrett-Crane model emerges as the limit of (3.4.4) for $\gamma \to \infty$:

$$\text{Barrett-Crane} = \lim_{\gamma\to\infty} \left(X^{IJ} - \gamma \, \epsilon_{KL}{}^{IJ} \, X^{KL} \right) n_J = 0 \tag{3.4.5}$$

$$\Rightarrow \left(\epsilon_{KL}{}^{IJ} \, X^{KL} \right) n_J = 0 \; . \tag{3.4.6}$$

In the above equation, if we use the decomposition for the Lie-algebra $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, we obtain that the two components of X_i^{IJ}, namely X_+^{IJ} and X_-^{IJ}, must have the same norm; this is indeed the condition stated in the previous section as the definition of the Barrett-Crane model.

Now let us focus on the solutions of (3.4.4) for finite γ, and let us consider two distinc cases, $\gamma > 1$ and $\gamma < 1$. Using the splitting (3.3.6), for $\gamma > 1$ the quantum version of the simplicity constraint requires the representations (j^+, j^-) of the two $\mathfrak{su}(2)$'s to be related by (for the details see [8])

$$j^+ = \frac{\gamma+1}{\gamma-1} \, j^- \tag{3.4.7}$$

where (j^+, j^-) are forced to be half integers. Recall that if we choose $j^+ = j^-$ then $\gamma \to \infty$, reducing EPRL/FK to the Barrett-Crane model, as already stated. The condition on the representation of the two $\mathfrak{su}(2)$'s to be expressed

with half integers values of j imposes a quantization rule on the allowed values for the Immirzi parameter γ, in fact it results:

$$\gamma = \frac{j^+ + j^-}{j^+ - j^-} \tag{3.4.8}$$

so γ must be rational. The case $\gamma < 1$ is solved as:

$$j^+ = \frac{\gamma + 1}{1 - \gamma}\, j^- \tag{3.4.9}$$

and the corresponding rational value for γ is

$$\gamma = \frac{j^+ - j^-}{j^+ + j^-} \quad . \tag{3.4.10}$$

These are the labels connected to each triangle in the 4-simplex $\wp$ used for the discretization of the space-time. As in the case of the Barrett-Crane model, we can substitute back this labellings of the representations of the two $\mathfrak{su}(2)$'s into equation (3.2.40), in order to obtain the generating functional for 4-dimensional BF-theories, taking into account the constraints imposed on the B-field to reduce to General Relativity.

Chapter 4

Introduction to the Group Field Theory

"There is nothing more practical
than a good theory."

K. Lewin

In this chapter we introduce the basic notions and ideas of Group Field Theory. This is a quantum field theory described in terms of field whose variables live in the gauge group G. The techniques and the tools used in developing the calculations are the same used in the common quantum field approach to gauge theories; an action is imposed with a term quadratic in the field (the kinetic energy) and a term, the potential, taking into account all the possible features and interactions of the field itself ([16] and [7]). Propagators, vertex amplitudes and Green's functions are calculated in the standard manner. The only difference is that all these fields are regarded as fields on a group manifold, and for this reason this method is particularly suited to describe the discretized models of quantum gravity ([21]); the main aim of this formulation is to find the spin foam amplitudes (as in the previous chapter) as Feynman diagrams of such a quantum field theory.

4.1 General formalism

In the context of Group Field Theory, a field is a map from the cartesian product of the gauge group into the complex numbers. If we consider Euclidean quantum gravity models, then $G = \mathrm{SO}(D)$, D being the dimension of the space-time manifold M; otherwise, for Lorentzian models, the gauge group will be the $G = \mathrm{SO}(D-1,1)$. Formally, we define

$$\begin{aligned} \phi : G^D \equiv G \times \ldots \times G &\longrightarrow \mathbb{C} \\ (g_1, \ldots, g_D) &\longrightarrow \phi(g_1, \ldots, g_D) \ . \end{aligned} \tag{4.1.1}$$

We can demand further properties on the form of $\phi(g_1, \ldots, g_D)$; first of all, it is required that the value assumed by ϕ in a point $(g_1, \ldots, g_D) \in G^D$ is invariant under a cyclic permutation of its arguments, i.e.

$$\phi(g_1, \ldots, g_D) = \phi(g_D, g_1, g_2, \ldots, g_{D-1}) = \phi(\ \pi(g_1, \ldots, g_D)\ ) \tag{4.1.2}$$

where π is a generic permutation of the elements $(g_1, \ldots, g_D)$. Now let us consider the right (or equivalently the left) action of the group G on itself, defined by

$$\begin{aligned} R_{g'} : G \times G &\longrightarrow G \\ (g', g) &\longrightarrow gg' \end{aligned} \tag{4.1.3}$$

We define the projector $\hat{P}_{g'}$ as:

$$\hat{P}_{g'}\ \phi(g_1, \ldots, g_D) = \int_G dg'\ \phi(g_1 g', \ldots, g_D g') \tag{4.1.4}$$

where dg' is the invariant Haar measure on the group G, normalized as $\int_G dg' = 1$. We require the full *gauge invariance* of the field ϕ in the form

$$\phi(g_1, \ldots, g_D) = \hat{P}_{g'}\ \phi(g_1, \ldots, g_D) \ . \tag{4.1.5}$$

Notice that the operator $\hat{P}_{g'}$ acts as a projector onto the right-translated space $G_{g'} \subseteq G$, which is the subset of G obtained by acting with g' to the right on each element $g \in G$. For this reason, the only eigenvalues allowed for $\hat{P}_{g'}$ are 0 and 1, $\forall\, g' \in G$.

The dynamics of the field is defined by imposing an action of the form [1]

$$\begin{aligned} S_D\left[\,\phi\,\right] &= \frac{1}{2}\int_{G^{2D}} dg_i\; dh_i\; \phi\left(g_i\right)\; \mathcal{K}\left(g_i, h_i\right)\; \phi\left(h_i\right) \;+ \\ &+ \frac{\lambda}{(D+1)!}\int_{G^{D(D+1)}} \prod_{i\neq j=1}^{D+1} dg_{ij}\; \mathcal{V}\left(g_{ij}\right) \;\times \\ &\times \phi\left(g_{1j}\right)\ldots\phi\left(g_{D+1j}\right) \end{aligned} \tag{4.1.6}$$

where $\phi\left(g_{1j}\right) \stackrel{def}{=} \phi\left(g_{12},\ldots,g_{1D+1}\right)$, i.e. all the possible arrangements of elements. $\mathcal{K}$ is the kinetic operator, whose inverse gives the propagator of the theory, and $\mathcal{V}$ is the potential operator, giving raise to the vertex function; the interaction is chosen to be of degree $D+1$, λ is a coupling constant. Specifying $\mathcal{K}$ and $\mathcal{V}$ we obtain different models.
According to the choice of the mathematical proprerties satisfied by the operators $\mathcal{K}$ and $\mathcal{V}$, one can require further symmetry properties on the action $S_D\left[\phi\right]$. Anyway, the action must surely be chosen to be gauge invariant:

$$S_D\left[\,\phi\,\right] = S_D\left[\;\hat{P}_{g'}\;\phi\;\right]\;. \tag{4.1.7}$$

If one refers to a specific group G (for instance we can take $G=\mathrm{SO}(D)$), the field ϕ can be expanded using the harmonic analysis of functions on that group, and can be rewritten in terms of normal modes; in this way, these expansion modes will be functions of group representations, instead of being functions of the group elements themselves.
Once an action for the field is provided, the techniques of the quantum field theory can be introduced. The generating functional assumes the form:

$$Z_G = \int_G \mathfrak{D}\left[\phi(g_i)\right]\; e^{i\;S_D[\phi]} \tag{4.1.8}$$

and it can be expressed, as usual, as an expansion in terms of powers of λ, each of them weighted with an appropriate Feynman amplitude (see [16])

$$Z_G = \sum_{\Gamma}\; \frac{\lambda^{v(\Gamma)}}{(D+1)!^{v}\; v!\;\left(\mathrm{sym}[\Gamma]\right)}\; Z\left[\Gamma\right] \tag{4.1.9}$$

where $v\left(\Gamma\right)$ is the number of vertices of the Feynman graph Γ; sym[Γ] refers to all the possible arrangements of legs and vertices in the graph Γ which

[1] We relax the notation and use $\phi\left(g_i\right)$ as $\phi\left(g_1,\ldots,g_D\right)$.

can be obtained by means of automorphisms of the diagram onto itself, that is, sym[Γ] is the number of topologically distinct Feynman diagrams with an assigned numbers of legs and vertices.
At this point we are interestend in finding the explicit form for the amplitude $Z[\Gamma]$ of a given Feynman graph, that is, the Feynman rules of the theory. The key result of Group Field Theory is that these amplitudes are exactly the same spin foam amplitudes we have found in the previous chapters; conversely, every spin foam model can be obtained as a Feynman diagram of a particular Group Field Theory action. This is because the field ϕ is a function on G^D (it "carries" D copies of the gauge group) and thus can be expanded in harmonic analysis on the group G, the normal modes and the coefficient depending on the representations of the group. In this way, one can substitute back this expansion into the action (4.1.6), so the Green's functions one calculates are then expressed in terms of representations functions; if one uses the Euclidean rotation group or the Lorentzian group, such functions can easily be expressed in terms of *spins*. Thus propagators and vertex functions can be written as a product of amplitudes, each one depending on a spin j ([7]). This form can be easily riarranged to give back the formula for a spin foam amplitude (2.2.7).

In the next section we are going to introduce some GFT models, in order to give a better understanding of the picture presented.

4.2 Group Field Theory models

According to the form of the operators $\mathcal{K}$ and $\mathcal{V}$, different GFT models can be obtained. Here we are going to present some models, related to particular choices of the above operators appearing in the action $S_D[\phi]$. At this point we focus our attention on a particular class of $\mathcal{K}$ and $\mathcal{V}$ operators. The kinetic operator $\mathcal{K}$ in D dimensions has $2D$ arguments: $\mathcal{K}(g_i, h_i)$. We choose it to be of the form:

$$\mathcal{K} = \prod_{i=1}^{D} \delta\left(g_i\, {h_i}^{-1}\right) \tag{4.2.1}$$

so that this ensures the kinetic energy in the action $S_D[\phi]$ to be always the field squared, with a number of arguments equal to the dimension of the space-time manifold, in the same spirit of the well-known quantum field theories. Furthermore, we choose the potential operator $\mathcal{V}$ as according to the following definition:

Definition. *A potential operator $\mathcal{V}$ is said to be simple if the Feynman vertex it originates with its propagators can be expressed as a product of δ functions, each δ function joining two different arguments of the propagators ingoing into the vertex.*

This definition can be better understood by means of a graphical representation, as in *Fig.* 4.1.

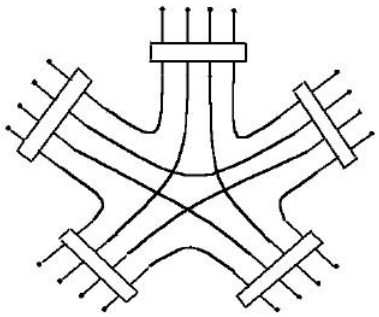

Figure 4.1: *A particular simple vertex in a* 4*-dimensional GFT model.*

It shows a simple vertex in a 4-dimensional Group Field Theory model. In four dimensions each propagator has four arguments, represented by the four parallel lines joining into the vertex from different angles. Each line of the propagator strands only to another one: this means that if we start from a line representing the argument g_i and we end to the argument h_j, we must introduce into the potential operator a δ function of the form $\delta\left(g_i\, h_j^{-1}\right)$. The product of all these δ functions defines the potential operator giving raise to such a vertex. From now on, we are going to consider only operators members of this classes.

The 2-dimensional model. Let us now restrict to the 2-dimensional case, i.e. the first one studied in the section § 3.2. Let us choose the gauge group

G to be $G = \mathrm{SU}(2)$. The action in the general form writes

$$\begin{aligned} S_2[\,\phi\,] &= \frac{1}{2}\int_{G^4} dg_1\, dg_2\, dh_1\, dh_2\ \phi(g_1,g_2)\ \mathcal{K}(g_1,g_2,h_1,h_2)\,\phi(h_1,h_2) + \\ &+ \frac{\lambda}{3!}\int_{G^6} dg_1\, dg_2\, dg_3\, dg_4\, dg_5\, dg_6\ \phi(g_1,g_2)\ \phi(g_3,g_4)\ \phi(g_5,g_6)\times \\ &\times\ \delta\left(g_1 g_3^{-1}\right)\ \delta\left(g_2 g_5^{-1}\right)\ \delta\left(g_4 g_6^{-1}\right) \end{aligned} \tag{4.2.2}$$

The kinetic operator $\mathcal{K}$, in the two-dimensional case, assumes the form:

$$\mathcal{K} = \delta\left(g_1 {h_1}^{-1}\right)\ \delta\left(g_2 {h_2}^{-1}\right) \tag{4.2.3}$$

and the action becomes, after carrying out the integration over the h_i variables

$$\begin{aligned} S_2[\,\phi\,] &= \frac{1}{2}\int_{SU(2)^2} dg_1\ dg_2\ \phi^2(g_1,g_2) + \\ &+ \frac{\lambda}{3!}\int_{SU(2)^3} dg_1\ dg_2\ dg_3\ \phi(g_1,g_2)\ \phi(g_2,g_3)\ \phi(g_1,g_3) \end{aligned} \tag{4.2.4}$$

so that each integration variable is shared by two fields. Now we use the harmonic analysis on $G = \mathrm{SU}(2)$, in order to decompose the field $\phi(g_i)$ into its Fourier modes. Using the same notations introduced in the previous (§ 3.2), let ρ be a representation of SU(2), V_ρ the vector space basis of the representation and D_ρ the group matrix element of such representation. Then, it can be shown (see [7]) that a function $\tilde{\phi}$ exists, such that the Fourier modes for the field ϕ can be written as:

$$\phi_\rho = \int_G dg\ \tilde{\phi}(g)\ D_\rho\left(g^{-1}\right) \tag{4.2.5}$$

where

$$\tilde{\phi}(g) = \sum_{\rho\in Irr\ rep(G)} \dim(\rho)\ tr[\,\phi_\rho D^\rho(g)\,]\ . \tag{4.2.6}$$

Inserting the expansions (4.2.5) and (4.2.6) into equation (4.2.4), by making use of the orthonormality condition for the matrix elements of the representations of SU(2) (see also the Appendix A.8)

$$\int_G dg\ D^j_{m,n}(g)\ \left(D^{j'}_{m',n'}(g)\right)^* = \frac{1}{\dim(\rho)}\ \delta^{jj'}\,\delta_{mm'}\,\delta_{nn'} \tag{4.2.7}$$

one obtains that the action reduces to:

$$S_2 [\ \phi\ ] = \sum_{\rho \in Irr\ \ rep(G)} \dim(\rho)\ \left(tr\left(\phi_\rho^2\right) + \frac{\lambda}{3!}\ tr\left(\phi_\rho^3\right) \right)\ . \tag{4.2.8}$$

Starting form this expression one can calculate the Feynman graph expansion in terms of functions of the representation ρ. This theory is the analogue of a ϕ^3 theory in the quantum field theory approach, because of the presence of the field ϕ three times into the expression of the potential appearing in the action. For this reason, the integration are carried twice for the propagator (i.e. the kinetic energy term) and three times for the vertex amplitude (i.e. the interaction term). This means that a Feynman diagram is composed out of propagators sharing two representations of the gauge group and vertices sharing three representations of the gauge group. This is exactly what happens with a 2-dimensional two complex, because in that case each vertex bounds three edges and each edge shares two faces. These are all the combinatorical elements required in a 2-dimensional spin foam; in this way, a Feynman diagram can be regarded as associated to a particular 2-complex on the space-time.
Moreover, if one completes the calculation for a diagram Γ, embedded into a manifold whose Euler characteristic is $\chi(M)$, one finds ([7] and [11]):

$$Z[\Gamma] = \sum_{\rho \in Irr\ \ rep(G)} \dim(\rho)^{\chi(M)} \tag{4.2.9}$$

which is exactly the same expression we found for the generating functional of a BF-theory in 2 dimensions, equation (3.2.18).

The 3-dimensional (Boulatov) model. This model is the generalization of the previous one in the case of 3 dimensions, and was first introduced by Boulatov, in [5]. The action (4.1.6) becomes, with the choices for the kinetic term as in (4.2.1) and for the potential term as a simple vertex operator (Boulatov vertex)

$$\begin{aligned} \mathcal{V}(g_1, \ldots, g_{12}) &= \delta\left(g_3\, {g_4}^{-1}\right)\ \delta\left(g_2\, {g_8}^{-1}\right)\ \delta\left(g_6\, {g_7}^{-1}\right)\ \delta\left(g_9\, {g_{10}}^{-1}\right) \times \\ &\times\ \delta\left(g_5\, {g_{11}}^{-1}\right)\ \delta\left(g_1\, {g_{12}}^{-1}\right) \end{aligned} \tag{4.2.10}$$

the following:

$$\begin{aligned} S_3[\,\phi\,] &= \frac{1}{2}\int_{SU(2)^3} dg_1\; dg_2\; dg_3\; \phi^2(g_1,g_2,g_3) \\ &+ \frac{\lambda}{4!}\int_{SU(2)^6} dg_1\; dg_2\; dg_3\; dg_4\; dg_5\; dg_6\; \phi(g_1,g_2,g_3)\;\times \\ &\times\; \phi(g_1,g_4,g_5)\;\phi(g_2,g_5,g_6)\;\phi(g_3,g_6,g_4)\quad . \end{aligned} \tag{4.2.11}$$

In three dimensions, using the relation

$$D^{(j)}_{m,n}(g) = \int_G dh\; D^{(j)}_{m,n}(g\,h) \tag{4.2.12}$$

which follows from

$$\int_G dh\; D^{(j)}_{m,n}(g\,h) = \int_G dh \sum_r\; D^{(j)}_{m,r}(g)\; D^{(j)}_{r,n}(h) = \sum_r\; D^{(j)}_{m,r}(g)\;\delta_{rn} \tag{4.2.13}$$

the field can be expanded as

$$\begin{aligned} \phi(g_1,g_2,g_3) &= \sum_{j_1,j_2,j_3}\sum_{\{m,n,k\}} \tilde{\phi}^{m_1,m_2,m_3;k_1,k_2,k_3}_{j_1,j_2,j_3}\; D^{j_1}_{m_1,n_1}(g_1)\; D^{j_2}_{m_2,n_2}(g_2)\;\times \\ &\times\; D^{j_3}_{m_3,n_3}(g_3)\int_G dh\; D^{j_1}_{n_1,k_1}(h)\; D^{j_2}_{n_2,k_2}(h)\; D^{j_3}_{n_3,k_3}(h) \end{aligned} \tag{4.2.14}$$

where $m_i, n_i, k_i = 1,\ldots,\dim(\rho)$. If we choose again the gauge group to be $G = \mathrm{SU}(2)$, the integral of the three matrix elements can be arranged in terms of product of Clebsch-Gordan coefficients, by using the Wigner $3j$-symbols also shown in the Appendix A.8:

$$\int_G dh\; D^{j_1}_{n_1,k_1}(h)\; D^{j_2}_{n_2,k_2}(h)\; D^{j_3}_{n_3,k_3}(h) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \tag{4.2.15}$$

where

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2+n_3}}{\sqrt{2j_3+1}}\;\langle j_1\,j_2;\; n_1\,n_2\,|\,j_1\,j_2;\; j_3\;-n_3\,\rangle\;. \tag{4.2.16}$$

One can substitute back these formulae into the action (4.2.11), in order to obtain an expression in terms of functions of the representations and Clebsch-Gordan coefficients. In this case, because of the presence of the field ϕ four times in the action (4.2.11), one can regard the Feynman graph obtained

as the analogue of the ϕ^4 diagrams in quantum field theory. We now have four quantities labelled with group representations, so we can think at them as to the faces of a tetrahedron, and in this way the Boulatov model can be regarded as a spin foam model in 3 dimensions. This is indeed true, because if one performs the calculation leading to the generating functional and to the Green's functions (we remand the reader to [5] for the full mathematical details), one obtains exactly the expression (3.2.33) we found for a 3-dimensional BF-theory on a lattice

$$Z_{[\mathcal{T},\mathcal{S}_{\mathcal{T}}]}(M) = \prod_{e\in\mathcal{T}} \sum_{\rho\in Irr\ rep(G)} (2j_e+1) \prod_{tetrahedra\in\mathcal{T}} \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\}_{tetrah} . \tag{4.2.17}$$

The measure $\mathfrak{D}[\phi(x)]$ in the path integral must be chosen as

$$\mathfrak{D}[\phi(x)] = \prod_{\{j,j',j''\}} \prod_{-j\leq m\leq j} dA^{m,m',m''}_{j,j',j''} \tag{4.2.18}$$

with

$$\begin{aligned} A^{m,m',m''}_{j,j',j''} &= \frac{1}{\sqrt{(2j+1)(2j'+1)(2j''+1)}} \times \\ &\times \sum_{k,k',k''} \tilde{\phi}^{m,m',m'',k,k',k''}_{j,j',j''} \begin{pmatrix} j & j' & j'' \\ k & k' & k'' \end{pmatrix} \end{aligned} \tag{4.2.19}$$

expressing linear combination of the Fourier modes $\tilde{\phi}^{m,m',m'',k,k',k''}_{j,j',j''}$. The sum in the (4.2.18) runs over all the (j,j',j'')'s obeying the triangle inequality $|j'-j''| \leq j \leq j'+j''$.

The 4-dimensional model. At this point it is easy to introduce the analogue model in four dimensions. This was first attempted in [15]. The key point is to regard a 4-dimensional GFT action as a ϕ^5 quantum field model, and so to write down the Feynman diagrams in a similar way. In four dimensions, the vertex operators looks exactly like as in *Fig.* 4.1, with five propagators ingoing into. The gauge group is still $G = \mathrm{SU}(2)$, and the field

can be expanded as:

$$\begin{aligned}\phi\left(g_1,g_2,g_3,g_4\right) &= \sum_{j_1,j_2,j_3,j_4}\sum_{\{m,n,k\}}\tilde{\phi}^{m_1,m_2,m_3,m_4;k_1,k_2,k_3,k_4}_{j_1,j_2,j_3,j_4}\times\\ &\times\ D^{j_1}_{m_1,n_1}\left(g_1\right)\ D^{j_2}_{m_2,n_2}\left(g_2\right)\ D^{j_3}_{m_3,n_3}\left(g_1\right)\ D^{j_4}_{m_4,n_4}\left(g_1\right)\times\\ &\times\ \int_G dh\ D^{j_1}_{n_1,k_1}\left(h\right)\ D^{j_2}_{n_2,k_2}\left(h\right)\ D^{j_3}_{n_3,k_3}\left(h\right)\ D^{j_4}_{n_4,k_4}\left(h\right)\end{aligned} \tag{4.2.20}$$

The integral of the product of four matrix elements of the representation of SU (2) is (see also the Appendix A.8)

$$\begin{aligned}\int_G dh\ \ D^{j_1}_{n_1,k_1}\left(h\right)\ D^{j_2}_{n_2,k_2}\left(h\right)\ D^{j_3}_{n_3,k_3}\left(h\right)\ D^{j_4}_{n_4,k_4}\left(h\right) &= \sum_{\substack{j,m,m'\\n,n'}}\begin{pmatrix}j_1 & j_2 & j\\ m_1 & m_2 & m\end{pmatrix}\times\\ &\times\ \begin{pmatrix}j & j_3 & j_4\\ m' & m_3 & m_4\end{pmatrix}\begin{pmatrix}j_1 & j_2 & j\\ n_1 & n_2 & n\end{pmatrix}\begin{pmatrix}j & j_3 & j_4\\ n' & n_3 & n_4\end{pmatrix}\\ &\times\ (-1)^{2j+m+n}\ (2j+1)\ \delta_{m,-m'}\ \delta_{n,-n'}\ .\end{aligned} \tag{4.2.21}$$

After carrying out the integration over the variable h, one can express the field $\phi\left(g_1,g_2,g_3,g_4\right)$ in terms of combination of the Wigner $3j$-symbols, which can be rearranged in order to give the $15j$-symbols. The generating functional is calculated by using a measure

$$\mathfrak{D}[\phi(x)] = \prod_{\{j,j_i\}}\ \prod_{\substack{-j_i\leq m_i\leq j_i\\ \sum_i m_i=0}} dM^{j_1,j_2;j;j_3,j_4}_{m_1,m_2,m_3,m_4} \tag{4.2.22}$$

with

$$\begin{aligned}M^{j_1,j_2;j;j_3,j_4}_{m_1,m_2,m_3,m_4} &= \sum_{\substack{n,n'\\ n_i,{n_i}'}}\tilde{\phi}^{m_1,m_2,m_3,m_4;k_1,k_2,k_3,k_4}_{j_1,j_2,j_3,j_4}\ \begin{pmatrix}j_1 & j_2 & j\\ {n_1}' & {n_2}' & n\end{pmatrix}\times\\ &\times\ \begin{pmatrix}j & j_3 & j_4\\ n' & {n_3}' & {n_4}'\end{pmatrix}\ (-1)^{j_1+j_2+j_3+j_4+n_1+n_2+n_3+n_4}\ (-1)^{j+n}\times\\ &\times\ \sqrt{(2j_1+1)\,(2j_2+1)\,(2j_3+1)\,(2j_4+1)\,(2j+1)}\times\\ &\times\ \delta_{n_1,-{n_1}'}\ \delta_{n_2,-{n_2}'}\ \delta_{n_3,-{n_3}'}\ \delta_{n_4,-{n_4}'}\ \delta_{n,-n'}\ .\end{aligned} \tag{4.2.23}$$

These formulae are the analogue of (4.2.18) and (4.2.19) in the four dimensional case. After performing the expansion of the field in terms of harmonic

analysis on the gauge group one should identify each term with a particular labelling associated to a 4-complex in the space-time. The generating functional assumes exactly the form (3.2.40) we found for a 4-dimensional BF-theory on a lattice.

Appendix A

Mathematical frameworks

"Geometry is the noblest branch of physics"

W. Osgood

This chapter is devoted to the introduction of some mathematical tools useful for a better understanding of the concepts developed in the previous sections. In particular we introduce the definitions of fibre bundles and principal bundles and the theory of connections on such bundles is presented in different ways. Particular attention is paid on the transformation properties of gauge potentials and field strenghts, as derived uniquely from the geometrical structure of the fibre bundle.

A.1 Definition of fibre bundles

Definition. *A* fibre bundle *is defined to be a set of elements* (E, π, M, F, G)*, consisting of*

1. *A differentiable manifold* E*, called the total space.*
2. *A differentiable manifold* M*, called the base space.*
3. *A differentiable manifold* F*, called the fibre.*
4. *A surjective projection map* $\pi : E \longrightarrow M$*. The inverse image of such a map,* $\pi^{-1}(m) \stackrel{def}{=} F_m,\ m \in M$*, is called the fibre at* $m \in M$*.*
5. *A Lie group* G *which acts on* F *by means of the left (right) action.*
6. *A covering of* M *with open sets* $\{U_i\}$*, equipped with a diffeomorphism* $\phi_i : U_i \times F \longrightarrow \pi^{-1}(U_i)$ *such that* $\pi \circ \phi_i(m, f) = m$*. We refer to the map* ϕ_i *as to the local trivialization of* U_i*, since it maps* $\pi^{-1}(U_i)$ *onto the direct product* $U_i \times F$*.*
7. *The map* $\phi_i(m, f)$ *can be regarded also as a map* $\phi_{i,m}(f)$*. On* $U_i \cap U_j \neq \emptyset$ *we require that* $t_{ij}(m) \stackrel{def}{=} \phi_{i,m}^{-1} \circ \phi_{j,m} : F \longrightarrow F$ *be an element of* G*. The maps* t_{ij} *are called the transition functions.*

Indeed, given a fibre bundle, the set of all possible transition functions t_{ij} is far from unique. If all the transition functions t_{ij} can be taken to be the identity maps, the fibre bundle is called *trivial*, because it can be expressed as the direct product $E = M \times F$. It can be shown that this condition is equivalent to the requirement that the base space M is contractible to a point; we do not give the proof of this statement here.

Given a fibre bundle, a global section $s : M \longrightarrow E$ is a differentiable map satisfying

$$\pi \circ s = id_M \ . \tag{A.1.1}$$

Obviously, taken $m \in M,\ s(m) \in \pi^{-1}(m) = F_m$, so that $(\pi \circ s)(m) = m$, $\forall\, m \in M$. It is not required for any fibre bundle to admit the existence of a global section satisfying (A.1.1) for all $m \in M$; in this case we may define (if possible) local sections satisfying (A.1.1) only on subsets of M. If

this subsets consist of the open covering [1] $\{U_i\}$ of M, we say that the fibre bundle admits a set of covering local sections. A fibre bundle is said to be *principal*, and denoted by $P(M,G)$, if the fibre F is the Lie group G.

A.2 Connections on principal bundles

Let $P(M,G)$ be a principal bundle whose fibre F is identical to the structure group G. Take $u \in P$ and let T_uP be the tangent space at u. The set of vertical vectors V_uP at u is defined as the subset of T_uP which is also tangent to the fibre F_m in $m = \pi(u)$.
According to this definition we explicitally construct the subspace V_uP. Let $A \in \mathfrak{g}$ be an element of the lie algebra $\mathfrak{g}$ of G; e^{tA} is an element of a one–parameter subgroup of G genereted by A. The right action of G on itself

$$G \times G \longrightarrow G$$
$$\left(u = \pi^{-1}(m) \in G_m = F_m \ , \ e^{tA} \right) \rightsquigarrow ue^{tA}$$

defines a curve through u belonging to the fibre, because $\pi(u) = \pi(ue^{tA})$. So any tangent vector defined by means of such a curve is tangent to G_m.
So we define the vector $A^{\#} \in V_uP$ by setting:

$$A^{\#} f(u) = \frac{d}{dt} \, f(\, ue^{tA} \,)_{|_{t=0}}$$

f being an arbitrary smooth function $\in \mathscr{F}(P)$. In this way a vector $A^{\#}$ is defined at each $u \in P$, so a vector field $A^{\#} \in \mathfrak{X}(P)$ can be constructed. It's easy to show the vector space isomorphism between $\mathfrak{g}$ and V_uP defined by $\#$

$$\# : \quad \mathfrak{g} \longrightarrow V_uP$$
$$A \rightsquigarrow A^{\#}.$$

Once the subspace V_uP is constructed, the horizontal subspace H_uP is determined as a complement of V_uP in T_uP.

[1] in the sense of the topology defined on M.

Theorem. *The set of vertical vectors V_uP could equivalently be defined as the kernel of the map π_* at $u \in P$, that is:*

$$X \in V_uP \iff \pi_* X = 0.$$

Proof. First note that the statement $\pi_* X = 0 \implies X \in V_uP$ is equivalent to $X \notin V_uP \implies \pi_* X \neq 0$. Then take $g \in \mathscr{F}(M)$:

$$\pi_* X [\, g \,] = X[\, g \circ \pi \,] = \frac{d}{dt} \left(g \circ \pi\right)(u(t))_{|_{t=0}} .$$

The derivative

$$\frac{d}{dt} \left(g \circ \pi\right)(u(t))_{|_{t=0}} = \frac{dg}{d\pi} \frac{d}{dt} \pi \left(u(t)\right) = \frac{dg}{d\pi} \frac{d}{dt} m(t)_{|_{t=0}} = 0 \iff m(t) = m_0$$

But the condition $\pi\left(u(t)\right) = m_0$ implies the curve $u(t)$ to belong to the fibre in m_0 for each value of t, so the tangent vector will act as a vector tangent to a curve in the fibre, i.e., by definition, as a vector of V_uP.
Now the second part of the theorem:

$$X^{\#} \in V_uP \implies \pi_* X^{\#} = 0.$$

Thus

$$\begin{aligned} \pi_* X^{\#} [\, g \,] &= X^{\#}[\, g \circ \pi \,] = 0 \qquad \forall\, g \in \mathscr{F}(M) \text{ if } X \in V_uP \\ X^{\#}[\, g \circ \pi \,] &= \frac{d}{dt} \left(g \circ \pi\right) \left(ue^{tX}\right)_{|_{t=0}} \quad \text{because } X \in V_uP \end{aligned}$$

But

$$\frac{d}{dt} \left(g \circ \pi\right) \left(ue^{tX}\right)_{|_{t=0}} = \frac{dg}{d\pi} \frac{d}{dt} \pi \left(ue^{tX}\right) = \frac{dg}{d\pi} \frac{d}{dt} \pi\left(u\right) = 0$$

because $X^{\#} \in V_uP$, that is the action of a flow generated by X leaves the fibre invariant.

□

Definition. *Let $P(M, G)$ be a principal bundle. A connection on P is a unique splitting of the tangent space T_uP such that the following conditions hold:*

- $T_uP = V_uP \ \oplus \ H_uP$

the assignment $u \to H_uP$ *depending smooth on* u*;*

- $\forall\, X \in T_uP \quad \Rightarrow \quad X = X^v + X^h \ ; \quad X^v \in V_uP,\ X^h \in H_uP$

- $H_{ug}P = \left(R_{g_*}\right)\ H_uP \qquad \forall\, u \in P,\ g \in G$

the last equality means that horizontal subspaces on the same fibre are related one other by the push forward map induced by the right action of the group on itself. So, given a horizontal subspace on a fibre, any horizontal subspace on the same fibre is constructed by right action.

In order to obtain such a splitting, a Lie–algebra valued one form $\omega \in \mathfrak{g} \otimes T^*P$ can be introduced. We want ω to be of the form:

- $\omega\left(A^{\#}\right) = A \quad (A \in \mathfrak{g}$ isomporphic to $V_uP)$

- $H_uP \stackrel{def}{=} Ker(\omega)$, that is $\{X \in T_uP : \omega(X) = 0\}$

Another property of ω has to be carried out, in order

$$H_{ug}P = \left(R_{g_*}\right)\ H_uP \qquad \forall\, u \in P,\ g \in G \tag{A.2.1}$$

to be satisfied. The property is the following

$$(R_g{}^*)\,\omega = Ad_{g^{-1}}\ \omega \tag{A.2.2}$$

in fact it can be shown that (A.2.2) implies (A.2.1).

Proposition. *If* (A.2.2) *holds then*

$$\left(R_{g_*}\right)\ H_uP = H_{ug}P \qquad \forall\, u \in P,\ g \in G$$

Proof. We first prove that $\left(R_{g_*}\right) H_uP \in H_{ug}P$. Let take $X \in H_uP$, then $\left(R_{g_*}\right) X$ belongs to $H_{ug}P$, in fact

$$\begin{aligned} \omega\left(R_{g_*}\ X\right) &= R_g{}^*\ \omega(X) \qquad \text{by definition of pullback map} \\ \omega\left(R_{g_*}\ X\right) &= R_g{}^*\ \omega(X) \ = \ Ad_{g^{-1}}\ \omega \ = \ g^{-1}\ \omega(X)\ g = 0 \end{aligned}$$

thus $\left(R_{g_*}\right) H_u P \in H_{ug} P$. Now it is to prove that if $Y \in H_{ug} P$, then there exists a vector $X \in H_u P$ such that

$$Y = \left(R_{g_*}\right) X$$

but this assertion widely holds because R_{g_*} is an invertible linear map.

□

A.3 Gauge potentials

Let $P(M, G)$ be a principal bundle. Let $\{U_i\}$ be an open covering of M with local section σ_i defined on each U_i [2].

Given a connection one form ω, a Lie–algebra–valued one form $\mathcal{A}_i$ on each U_i is automatically defined by putting

$$\mathcal{A}_i = {\sigma_i}^* \ \omega \ \in \ \mathfrak{g} \otimes \Omega^1(M)$$

which depends on the section σ_i. We refer to the $\mathcal{A}_i$'s as to the gauge potentials defined on M by ω, once a covering atlas (U_i, σ_i) is assigned on M.

Now a question automatically arises. It's important to specify if, conversely, the knowledge of the gauge potentials on M uniquely identifies the connection one form ω on P. In order to answer this question, we explicitally construct such a form, and show that it is the connection form required.

To construct ω explicitally it's necessary to introduce the canonical local trivialization on a fibre bundle. Let $P(M, G)$ be a principal bundle with a set of open covering $\{U_i\}$ of M, then consider the following:

[2] In the following we also refer to the couple (U_i, σ_i) as to a covering atlas of M.

Definition. *Let take the map*

$$\phi_i : \quad U_i \ \times \ F \quad \longrightarrow \quad \pi^{-1}\,(U_i)$$

$$(m, f) \quad \rightsquigarrow \quad u$$

such that

$$\pi \circ \phi_i\,(m, f) = m$$

that is

$$U_i \ \times \ F \quad \xrightarrow{\phi_i} \quad \pi^{-1}\,(U_i) \quad \xrightarrow{\pi} \quad U_i$$

$$(m, f) \qquad \rightsquigarrow \qquad m \quad .$$

The map ${\phi_i}^{-1}$ is called the canonical local trivialization of $\pi^{-1}(U_i)$, because it maps $\pi^{-1}(U_i)$ onto the direct product $U_i \ \times \ F$

$${\phi_i}^{-1} : \quad \pi^{-1}\,(U_i) \quad \longrightarrow \quad U_i \ \times \ F$$

so it splits locally the principal bunlde $P(M, G)$.

On the intersection $U_i \cap U_j \neq 0$ the map

$$t_{ij}(m) \stackrel{def}{=} {\phi_i}^{-1} \ \circ \ \phi_j : \ U_j \ \times \ F \quad \longrightarrow \quad U_i \ \times \ F$$

is required to be smooth and, once $m \in M$ is fixed, to belong to G. The t_{ij} are referred to be the transition functions from $U_j \ \times \ F$ to $U_i \ \times \ F$.

A fibre bundle is called trivial if it exists a trivialization of the form

$$\pi^{-1}\,(U_i) \quad \longrightarrow \quad U_i \ \times \ F$$

which acts globally on M, i.e. $\exists$ an application

$$P \quad \longrightarrow \quad M \ \times \ F$$

In that case it is a common use to identify P itself with the cartesian product $M \ \times \ F$. It can be shown that such an application always exists if we restric all the transition functions t_{ij} to be the identity map; this can be made only if the base space M is contractible to a point.

Theorem. *Given an open covering atlas (U_i, σ_i) on M and a set of gauge potentials $\mathcal{A}_i$, then*

$$\omega_i = {g_i}^{-1}\pi^*\mathcal{A}_i \; g_i + {g_i}^{-1} \; dg_i$$

is a connection one form on P if $g_i \in G$ is the canonical local trivialization on $\pi^{-1}(U_i)$ defined by

$$\phi_i{}^{-1}(\; u = \sigma_i(m)g_i \;) = (m, g_i)$$

Proof. We first prove that

$$\sigma_i{}^* \; \omega_i = \mathcal{A}_i \quad .$$

Take $X \in T_mM$ and evaluate

$$(\sigma_i{}^*\omega_i)\,(X) = \omega(\sigma_{i*}X) = {g_i}^{-1} \; \pi^*\mathcal{A}_i(\sigma_{i*}X) \; g_i + {g_i}^{-1} \; dg_i(\sigma_{i*}X)$$

If $\mathcal{A}_i$ is defined on $m \in M \Rightarrow \sigma_i{}^*\omega_i \in T_m^*M$, then $u \in P$ must be $\sigma_i(m)$. But in the definition of local trivialization we put $u = \sigma_i(m)g_i$, so it follows that the g_i's must be the identity element if we act on ω with the pullback of σ_i. It follows that

$$\begin{aligned}(\sigma_i{}^*\omega_i)\,(X) &= \pi^*\mathcal{A}_i(X) + dg_i(\sigma_{i*}X) \\ (\sigma_i{}^*\omega_i)\,(X) &= \mathcal{A}_i(\pi_* \; \sigma_{i*}X) + dg_i(\sigma_{i*}X)\end{aligned}$$

Note that $\pi_* \; \sigma_{i*} = id_{T_mM}$ by definition of local section. Moreover the element g_i is forced to be the identity on all the flow carried by σ_{i*}, so it cames out that

$$(\sigma_i{}^*\omega_i)\,(X) = \mathcal{A}_i(X) \qquad \forall\, X \in T_mM$$

that is

$$\sigma_i{}^*\omega_i = \mathcal{A}_i \; .$$

Now the proof ought to go on by showing that ω really satisfies the property required to be a correct connection one form on a fibre bundle, but this part of the proof will be not given here.

□

We have constructed ω with a pullback on each U_i on M, but in order ω to be uniquely defined on P, i.e. in order the splitting $T_uP = V_uP \; \oplus \; H_uP$ to be unique, as required in the definition of the connection, it must be

$$\omega_i = \omega_j \;\; \text{on} \;\; U_i \cap U_j \neq 0 \tag{A.3.1}$$

By means of the properties of transformation of the push forward map σ_{i*} of the local sections and of the definition of the transition functions t_{ij}, it can be shown that condition (A.3.1) is satisfied if and only if the gauge potentials, as seen in two different charts, transform as

$$\mathcal{A}_j = t_{ij}^{-1}\ \mathcal{A}_i\ t_{ij} + t_{ij}^{-1}\ dt_{ij} \tag{A.3.2}$$

So the gauge potentials as seen in different charts must be related by (A.3.2) in order to give raise to a well–defined connection one form on P. It means that a singular $\mathcal{A}_i$ cannot contain all the information on the separation of the bundle, but is the set of all $\mathcal{A}_i$'s related by (A.3.2) to carry out such information.

Example: the $U(1)$ bundle. Let $P(M, U(1))$ be a principal bundle. The structure group $U(1)$ is the circle S^1 and its Lie algebra is a line spanned by the element $i \in \mathbb{C}$. Let a covering atlas (U_i, σ_i) be given on M, such that $U_i \cap U_j \neq 0$; $\mathcal{A}_i$ are gauge potentials as seen in each chart on M. According to (A.3.2) they must be related in such a way

$$\mathcal{A}_j = t_{ij}^{-1}\ \mathcal{A}_i\ t_{ij} + t_{ij}^{-1}\ dt_{ij}$$

where $t_{ij,m} \in U(1)$ is forced to be of the form $e^{if(m)}$. So

$$\begin{aligned} \mathcal{A}_j &= e^{-if(m)}\ \mathcal{A}_i\ e^{if(m)} + e^{-if(m)} e^{if(m)}\ i\ df(m) \\ \mathcal{A}_j &= \mathcal{A}_i + i\ df \end{aligned} \tag{A.3.3}$$

$\mathcal{A}_j$ by definition $\in \mathfrak{u}(1) \times \Omega^1(M)$, so it can be decomposed using a base on $\mathfrak{u}(1)$; then $\mathcal{A}_j = i\ A_j$, where A_j is a real valued one form $\in \Omega^1(M)$. It follows that

$$A_j = A_i + df \tag{A.3.4}$$

is the transformation law admitted for the component $\Omega^1(M)$ of the gauge one form.

A.4 Covariant exterior derivative and curvature in principal bundles

Given a vector space V and a manifold M, a completely antisimmetric map

$$\omega : \underbrace{TM \times \ldots \times TM}_{r \text{ times}} \longrightarrow V$$

is said to be a vector valued r-form. It's easy to see that ω can always be decomposed in the form

$$\omega = \phi^{\alpha} \otimes \underline{e}_{\alpha}$$

$\{\underline{e}_{\alpha}\}$ being a basis of V, $\phi^{\alpha} \in \Omega^{r}(M)$.

Definition. *Let $P(M,G)$ be a principal bundle whit a connection one form separating T_uP into $V_uP \oplus H_uP$. The covariant exterior derivative of a vector valued r form is defined as:*

$$D : \Omega^{r}(M) \times V \longrightarrow \Omega^{r+1}(M) \times V$$

such that

$$D\omega\ (X_1, \ldots, X_{r+1}) = d\omega\ ({X_1}^h, \ldots, {X_{r+1}}^h)$$

where $X_i \in T_uP$ can be decomposed into $X = X^v + X^h$.

Definition. *The covariant extrerior derivative of the connection one form ω*

$$\Omega = D\omega$$

is said to be the curvature two form of the bundle generated by ω.

It can be shown that, given $X, Y \in T_uP$, the curvature two form satisfies

$$\Omega\ (X,Y) = d\omega\ (X,Y) + [\omega(X), \omega(Y)] \tag{A.4.1}$$

known as the Cartan's structure equation. (A.4.1) can be also put in the form

$$\Omega = d\omega + \omega \wedge \omega \quad . \tag{A.4.2}$$

Let the manifold M be equipped with a covering atlas (U_i, σ_i); by considering the definition of gauge potential as the local form of the connection ω, it is obvious that an expression such

$$\mathcal{F}_i = {\sigma_i}^* \ \Omega \tag{A.4.3}$$

must give the local form (or field strenght) on a chart U_i of the curvature two form. Let's explicitally write (A.4.3), we get

$$\begin{aligned} \mathcal{F}_i &= {\sigma_i}^* \ (d\omega + \omega \wedge \omega) = {\sigma_i}^*(d\omega) + {\sigma_i}^*\omega \wedge {\sigma_i}^*\omega \\ \mathcal{F}_i &= d({\sigma_i}^*\omega) + {\sigma_i}^*\omega \wedge {\sigma_i}^*\omega \\ \mathcal{F}_i &= d\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i \end{aligned} \tag{A.4.4}$$

which is the local expression of $\mathcal{F}_i$ in terms of gauge potentials $\mathcal{A}_i$ on a chart U_i. It's useful to write (A.4.4) by using a set of coordinate $x^\mu = (\varphi(m))^\mu$ on a chart U_i

$$(\mathcal{F}_i)_{\mu\nu} = \partial_\mu \, (\mathcal{A}_i)_\nu - \partial_\nu \, (\mathcal{A}_i)_\mu + [\ (\mathcal{A}_i)_\mu \, , (\mathcal{A}_i)_\nu \ ]$$

Since $(\mathcal{A}_i)_\mu$ and $(\mathcal{F}_i)_{\mu\nu}$ are both $\mathfrak{g}$–valued functions, they can be expanded using a base of $\mathfrak{g}$, so:

$$(\mathcal{F}_i)_{\mu\nu} = (F_i)^\alpha_{\mu\nu} \ \underline{e}_\alpha \qquad\qquad (\mathcal{A}_i)_\mu = (A_i)^\alpha_\mu \ \underline{e}_\alpha$$

to obtain

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + c_{\beta\gamma}^{\ \alpha} \ A^\beta_\mu A^\gamma_\nu \tag{A.4.5}$$

where $[\ \underline{e}_\beta, \underline{e}_\gamma \ ] = c_{\beta\gamma}^{\ \delta} \ \underline{e}_\delta$ is the Lie bracket of the basis elements of $\mathfrak{g}$.

In section 2 it has been deduced that, in order ω to be uniquely defined by means of the $\mathcal{A}_i$'s on (U_i, σ_i), the $\mathcal{A}_i$'s are forced to transform as

$$\mathcal{A}_j = t_{ij}^{-1} \ \mathcal{A}_i \ t_{ij} + t_{ij}^{-1} \ dt_{ij}$$

on $U_i \cap U_j \neq 0$. Inserting this condition into (A.4.4) it's easy to obtain the relation between the local curvature two form as seen in different charts:

$$\mathcal{F}_j = t_{ij}^{-1} \ \mathcal{F}_i \ t_{ij} \quad . \tag{A.4.6}$$

Theorem (Bianchi)**.** *Given a curvature two form* Ω*, then*

$$D\Omega = 0 \quad .$$

Proof. Take $X, Y, Z \in T_u P$, then

$$\begin{aligned} D\Omega(X,Y,Z) &\stackrel{def}{=} d\Omega\left(X^h, Y^h, Z^h\right) \\ D\Omega(X,Y,Z) &= d\left(d\omega + \omega \wedge \omega\right)\left(X^h, Y^h, Z^h\right) \\ D\Omega(X,Y,Z) &= \left(d\omega \wedge \omega + (-1)^{deg\ \omega}\ \omega \wedge d\omega\right)\left(X^h, Y^h, Z^h\right) = 0 \end{aligned}$$

because $\omega(X^h) = 0$ by definition of X^h.

□

By acting with $\sigma_i{}^*$ on $D\Omega$ we obtain the local form of Bianchi identity as written on a chart (U_i, σ_i):

$$\mathcal{D}\mathcal{F}_i = \sigma_i{}^*\left(D\Omega\right) = d\mathcal{F}_i + [\mathcal{A}_i, \mathcal{F}_i] = 0 \quad .$$

To conclude this section we note that if the Lie algebra $\mathfrak{g}$ is one dimensional, then relation (A.4.4) can be written

$$(\mathcal{F}_i)_{\mu\nu} = (F_i)_{\mu\nu}\ \underline{u} \qquad (\mathcal{A}_i)_{\mu} = (A_i)_{\mu}\ \underline{u}$$

so we can omitt the supercript α in writing the component expression and get

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{A.4.7}$$

provided $c_{\beta\gamma}^{\alpha}$ to be identically zero. Furthermore in the particular example of $U(1)$ bundles, we find that the law of transformation allowed for the field strenght is

$$\mathcal{F}_j = \mathcal{F}_i$$

so, in different charts of a (1) bundle, the curvature two form possesses the same expression; its components are related to the gauge potential by means of expression (A.4.7).

A.5 Covariant derivative of vector fields

In this section we introduce the notion of horizontal lift of a path and define a covariant derivative for vector fields depending on the connection one form defined on $P(M, G)$.

Definition. *Let $\gamma : [0,1] \to M$ be a curve on M. A curve $\widetilde{\gamma} : [0,1] \to P$ is said to be the horizontal lift of γ if*

$$\pi \circ \widetilde{\gamma} = \gamma$$

and

$$X_{|_{\widetilde{\gamma}(t)}} \in H_{\widetilde{\gamma}(t)} P \quad .$$

The second condition is necessary in order the curve to be uniquely defined (Cauchy's theorem on ordinary differential equation needs a condition on the tangent vectors). The following result holds:

Theorem. *Let $\gamma : [0,1] \to M$ be a curve on M, with $u_0 \in \pi^{-1}(\gamma(0))$. Then there exists a unique horizontal lift $\widetilde{\gamma}$ such that $\widetilde{\gamma}(0) = u_0$.*

It's easy to mind that such a theorem is proved by using the foundamental theorem on ordinary differential equation that assures the local existence and uniqueness of the solution, once an "initial condition" is specified. Note that the horizontal lift depends on the connection one form through the condition

$$X_{|_{\widetilde{\gamma}(t)}} \in H_{\widetilde{\gamma}(t)} P$$

which means $\omega(X) = 0$ for any tangent vector to the curve $\widetilde{\gamma}$.

Now consider a curve $\gamma : [0,1] \to M$ and take a point $u_0 \in \pi^{-1}(\gamma(0))$. The horizontal lift of γ provides a curve $\widetilde{\gamma}$ such that $\widetilde{\gamma}(0) = u_0$. Because of the uniqueness of such a curve, then there exists a unique $u_1 = \widetilde{\gamma}(1) \in \pi^{-1}(\gamma(1))$. So $\widetilde{\gamma}$ provides a corrispondence between

$$u_0 \quad \longrightarrow \quad u_1$$

u_1 refered as the parallel transport of u_0 along $\widetilde{\gamma}$. In general a map can be introduced

$$\Gamma_{\widetilde{\gamma}} : \quad [0,1] \times P \quad \longrightarrow \quad P \tag{A.5.1}$$

$$(t, u_0) \quad \rightsquigarrow \quad u_t = \Gamma_{\widetilde{\gamma}}(u_0) \stackrel{def}{=} \widetilde{\gamma}(t)$$

that associates to each point u_0 a unique point u_t going through the curve $\widetilde{\gamma}$. It's important to stress that this assignement depend on the choice of horizontal lift, i.e. on the choice of ω. The map $\Gamma_{\widetilde{\gamma}}$ provides a rule of transport of points $\in P$ once a curve on M is taken. This rule can be used to associate a unique point u_t to each point u_0 in a way depending on the connection. If we choose a loop on M, i.e. a curve such $\gamma(0) = \gamma(1)$, the points u_0 and its parallel transport $u_1 = \widetilde{\gamma}(1)$ must lie on the same fibre $\pi^{-1}(\gamma(0))$, but need not to be the same point. So any loop $\mathcal{C}$ in M provides couples

$$(u_0, u_1)_{\mathcal{C}}$$

related by a transformation $\tau_{\mathcal{C}}$, $u_1 = \tau_{\mathcal{C}} u_0$. The set of all $\tau_{\mathcal{C}}$, $\mathcal{C}$ being a loop through $m_0 = \pi(u_0)$, is a subgroup of G and is identified with the holonomy group at u_0 generated by ω.

We are now able to define a rule of transport for vectors, depending on the connection form ω, that enables the definition of a covariant derivative. Let $P(M, G)$ be a tangent bundle on M. The elements of the total space TM are tangent vectors to the manifold M. If a curve on M is given, a law of transformation for points in M is defined; but by means of horizontal lift a rule of transport is provided also for elements in TM:

$$\gamma : m \quad \longrightarrow \quad m_t$$

$$\widetilde{\gamma} : Y \quad \longrightarrow \quad Y_t$$

So we can give sense to the expression

$$\lim_{t \to 0} \frac{1}{t} \left(\overline{Y}_{|_{m_t}} - Y_{|_{m_0}} \right)$$

where $\overline{Y}_{|_{m_t}}$ must belong to the same fibre of $Y_{|_{m_0}}$ in order to be compared. Such expression get a sense by putting

$$\overline{Y}_{|_{m_t}} = (\Gamma_{\widetilde{\gamma},t})^{-1} \, Y_{|_{m_t}}$$

so we obtain

$$Y'_{|_{m_0}} = \lim_{t \to 0} \frac{1}{t} \left[\, (\Gamma_{\widetilde{\gamma},t})^{-1} \, Y_{|_{m_t}} - Y_{|_{m_0}} \right] \tag{A.5.2}$$

where $\Gamma_{\tilde{\gamma}}$ is the rule of transport associated to the curve γ by means of a connection form ω on TM. If γ is a flow generated by the field X, we refer to the previous expression as to the covariant derivative of the vector field Y through X, evaluated at m_0. By changing the point of evaluation m_0 we get a vector field corrispondence:

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \quad \longrightarrow \quad \mathfrak{X}(M)$$

$$(X, Y) \quad \rightsquigarrow \quad \nabla_X Y$$

Such a corrispondence defines an operator, called the covariant derivative ∇_X, which acting on Y gives raise to $\nabla_X Y$. This operator satisfies

- linearity in X and Y
- $\nabla_X (fY) = f\ \nabla_X Y + (L_X f)\, Y$ Leibniz rule on $f \in \mathscr{F}(M)$
- $\nabla_{(fX)} Y = f\ \nabla_X Y$ invariance under reparametrization of the field

the last relation being very useful in employing the covariant derivative as a correct derivation rule for physical field theory.

A.6 Other definition of connection

In this section we present a more general definition of connection on a bundle, which does not involve the structure group G and its action on a fibre of P. This way of introducing connections is based on a particular choice of a "projectable field" in a sense that we are going to specify.

Definition. *Let $X \in \mathfrak{X}(P)$ be a vector field on P; X is said to be vertical if, for any $f \in \mathscr{F}(M)$, it happens that*

$$L_X (\pi^* f) = 0.$$

We notice that this equation means

$$L_X (\pi^* f) = L_X (f \circ \pi) = X [f \circ \pi] = 0$$

but this condition must hold for any point $u \in P$, so if we evaluate the previous expression at $u \in P$ it holds

$$\left(X\,[f \circ \pi]\right)_{|_u} = X_{|_u}\,[f \circ \pi] = \left(\pi_* X_{|_u}\right)[\ f\ ] = 0$$

which means that $X_{|_u}$ belongs to the kernel of π_* at u. This definition of vertical field is then agree with the definition given in Section 1, which introduces the vertical spaces $V_u P$ as the kernel of the map π_* at each point $u \in P$.

Definition. *A field $Y \in \mathfrak{X}(P)$ is said to be projectable if $\exists\ X \in \mathfrak{X}(M)$ such that, for any $f \in \mathscr{F}(M)$*

$$L_Y\,(\pi^* f) = \pi^*\,(L_X f)\ .$$

In that case we refer to the field Y as to $X^{\uparrow}$, and the previous relation may be written as

$$L_{X^{\uparrow}}\,(\pi^* f) = \pi^*\,(L_X\ f)$$

and it seems that the operator $\uparrow$ "brings" X to act on f before π^*.
With this definition we can regard a vertical field as a field which projects on the null field in $\mathfrak{X}(M)$. The following proposition shows that the set of vertical fields $\mathfrak{X}^v(P)$ in an ideal in $\mathfrak{X}^\pi(P)$.

Proposition. *The set of projectable fields could be equivalently defined as the set of fields which leaves $\mathfrak{X}^v(P)$ invariant under the action of the Lie bracket, that is, given $Z \in \mathfrak{X}^v(P)$*

$$Y \in \mathfrak{X}^\pi(P) \iff [Z, Y] \in \mathfrak{X}^v(P)$$

Proof. First we prove that

$$Y \in \mathfrak{X}^\pi(P) \implies [Z, Y] \in \mathfrak{X}^v(P)\ .$$

Because Y is projectable then there exists a field $X \in \mathfrak{X}(M)$ such that

$$\begin{aligned} L_Y\,(\pi^* f) &= \pi^*\,(L_X\ f) && \forall\, f \in \mathscr{F}(M) \\ L_Z\ L_Y\,(\pi^* f) &= L_Z\ \pi^*\,(L_X\ f) = 0 \end{aligned}$$

because Z is vertical. We can add a null contribute to this equation without violating the identity, and we choose it of the form $-\ L_Y\ L_Z\,(\pi^* f)$. So we get

$$L_Z\ L_Y\,(\pi^* f) - L_Y\ L_Z\,(\pi^* f) = L_{[Z,Y]}\,(\pi^* f) = 0$$

which means that $[Z, Y]$ is a vertical field. Conversely, if $[Z, Y]$ is a vertical field then

$$L_{[Z,Y]}\,(\pi^* f) = 0 \qquad\qquad \forall\, f \in \mathscr{F}(M)$$

$$L_Z\ L_Y\,(\pi^* f) - L_Y\ L_Z\,(\pi^* f) = L_Z\ L_Y\,(\pi^* f) = 0\ .$$

Z is a vertical field for hypothesis, so this condition implies $L_Y\,(\pi^* f)$ to be of the form $\pi^*(\ )$, i.e. Y must be projectabile on a particular field in $\mathfrak{X}(M)$.

□

Indeed it is obvious that if we fix $X \in \mathfrak{X}(M)$ there exist different $Y \in \mathfrak{X}(P)$ such that

$$Y = X^{\uparrow}$$

in fact we can choose a particular $X^{\uparrow}$ and add a generical vertical field to obtain another field which projects on the same $X \in \mathfrak{X}(M)$. We say the lift of a vector field $X \in \mathfrak{X}(M)$ is determined up to vertical fields. The particular choice of the lift $X^{\uparrow}$, once X is assigned, defines the choice of connection. Let construct a map

$$\begin{array}{rccc} \sigma: & \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}^{\pi}(P) \\ & X & \rightsquigarrow & X^{\uparrow} \end{array} \tag{A.6.1}$$

which is required to be an $\mathscr{F}(M) - module$, that is

$$(f\ X)^{\uparrow} = (\pi^* f)\, X^{\uparrow}$$

further require

$$\pi_* \circ\ \sigma = id_{\mathfrak{X}(M)}\ .$$

Under this assumption we call the map (A.6.1) a connection on the bundle $P(M, G)$. Notice that this definition of connection does not involve the structure group G and its action on the fibre but it just makes use of the spaces P and M.

Definition. *The set of vector fields spanned as a module on P by the lift $X^{\uparrow}$ are defined to be the horizontal fields $\mathfrak{X}^h(P)$.*

The map (A.6.1) need not to be a Lie–algebra homomorphism. The failure of this condition is taken into account by the quantity

$$\Omega\left(X_1, X_2\right) \stackrel{def}{=} [X_1, X_2]^{\uparrow} - [X_1^{\uparrow}, X_1^{\uparrow}]$$

named the curvature of the connection.

Proposition.

$$\Omega\left(X_1, X_2\right) \in \mathfrak{X}^v(P)$$

Proof. Take $f \in \mathscr{F}(M)$ and evaluate

$$\begin{aligned}
L_{\Omega}\left(\pi^* f\right) &= L_{[X_1,X_2]^{\uparrow}-[X_1^{\uparrow},X_1^{\uparrow}]}\left(\pi^* f\right) \\
L_{\Omega}\left(\pi^* f\right) &= L_{[X_1,X_2]^{\uparrow}}\left(\pi^* f\right) - L_{[X_1^{\uparrow},X_1^{\uparrow}]}\left(\pi^* f\right) \\
L_{\Omega}\left(\pi^* f\right) &= \pi^*\left(L_{[X_1,X_2]}\ f\right) - \left(L_{X_1^{\uparrow}}\ L_{X_2{}^{\uparrow}} - L_{X_2^{\uparrow}}\ L_{X_1{}^{\uparrow}}\right)\left(\pi^* f\right) \\
L_{\Omega}\left(\pi^* f\right) &= \pi^*\left(L_{[X_1,X_2]}\ f\right) - \pi^*\left(L_{[X_1,X_2]}\ f\right) = 0
\end{aligned}$$

how can be easily evaluated by means of the definition of $X^{\uparrow}$.

□

The map π between P and M induces a pullback of forms at each point of M. So let's now consider the pullback $\pi^*\Omega$ evaluated in a point $m = \pi(u) \in M$ and calculate its action on vertical vectors:

$$\left(\pi^*\Omega\right)\left(Y_1, Y_2\right) \quad \text{with } Y_1, Y_2 \in V_u P$$

$$\left(\pi^*\Omega\right)\left(Y_1, Y_2\right) = \Omega\left(\pi_* Y_1, \pi_* Y_2\right)$$

this relation is seen to be vanishing, because vertical vectors $\in V_u P$ are in the kernel of π_* at each point $u \in P$; in this sense we often refer to this condition by saying the curvature is "horizontal" (note the abuse: indeed the curvature is a vertical field because its action on fields in $\mathfrak{X}(M)$ gives raise to a field in $\mathfrak{X}^v(P)$; we say it is "horizontal" in the sense that its pullback at each point $\pi^*\Omega$ vanishes if evaluated on vertical vectors).
This behavior can be compared with the same behavior of the curvature form

defined in Section 3 as the covariant exterior derivative of the connection one form; such an application

$$\underline{\Omega} \ \in \Omega^2(P) \otimes \mathfrak{g}$$

also vanishes on vertical vectors $\in V_u P$ at each point $u \in P$, because it is defined as an horizontal exterior derivative. So if we could identify the sets $\mathfrak{g}$ and $\mathfrak{X}^v(P)$ the two different maps we introduced in the description of a connection theory, named "curvature", could be regarded as the same object as descripted by using different formalism. Now we show how to do this identification.

Let's take a basis $(V_1, \ldots, V_n)$ in $\mathfrak{g}$. Each basis vector V_i generates a one-parameter subgroup $g_i(t) = e^{tV_i} g(0)$ on G. By the right action on the manifold P, this induces a flow $\sigma(t) \in P$; the map that associates to each point on P the tangent vector to the flow induced by G is called the induced vector field $V_i^{\#}$ generated by $V_i \in \mathfrak{g}$. The corrispondence

$$\# : \quad \mathfrak{g} \quad \longrightarrow \quad \mathfrak{X}(P)$$

is an isomorphism. The set of induced vector fields $V_i^{\#}$ span as a module the entire set $\mathfrak{X}^v(P)$; in this sense we can identify $\mathfrak{g}$ and $\mathfrak{X}^v(P)$, up to an isomorphism $\#$.

A.7 List of nj-symbols

In this appendix we want to list the explicit form of all the nj-symbols used in the previous chapters, so that the reader may found it easier to solve the explicit form of some generating functional given in terms of these combinations of Clebsch-Gordan coefficients.

The Wigner $3j$-symbols.

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} = \frac{(-1)^{j_1 - j_2 + n_3}}{\sqrt{2j_3 + 1}} \, \langle j_1 \, j_2; \, n_1 \, n_2 \, | \, j_1 \, j_2; \, j_3 \, - n_3 \, \rangle \, . \qquad \text{(A.7.1)}$$

The Racah-Wigner $6j$-symbols

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \sum_{-j_i \leq m_i \leq j_i} (-1)^{j_4+j_5+j_6+m_4+m_5+m_6} \times$$
$$\times \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_5 & j_6 & j_1 \\ m_5 & -m_6 & m_1 \end{pmatrix} \times$$
$$\times \begin{pmatrix} j_6 & j_4 & j_2 \\ m_6 & -m_4 & m_2 \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_3 \\ m_4 & -m_5 & m_3 \end{pmatrix} \quad \text{(A.7.2)}$$

The $15j$-symbols.

$$\begin{Bmatrix} l_1 & l_2 & l_3 & l_4 & l_5 \\ j_1 & j_2 & j_3 & j_4 & j_5 \\ l_{10} & l_9 & l_8 & l_7 & l_6 \end{Bmatrix} = \sum_{m,n} (-1)^{\sum_{i=1}^{10}(j_i+m_i)} (-1)^{\sum_{i=1}^{5}(l_i+n_i)} \times$$
$$\times \begin{pmatrix} j_1 & l_1 & l_2 \\ m_1 & n_1 & -n_2 \end{pmatrix} \begin{pmatrix} j_2 & l_2 & l_3 \\ m_2 & n_2 & -n_3 \end{pmatrix} \begin{pmatrix} j_3 & l_3 & l_4 \\ m_3 & n_3 & -n_4 \end{pmatrix} \times$$
$$\times \begin{pmatrix} j_4 & l_4 & l_5 \\ m_4 & n_4 & -n_5 \end{pmatrix} \begin{pmatrix} j_5 & l_5 & l_6 \\ m_5 & n_5 & -n_6 \end{pmatrix} \begin{pmatrix} j_2 & l_6 & l_7 \\ -m_2 & n_6 & -n_7 \end{pmatrix} \times$$
$$\times \begin{pmatrix} j_1 & l_7 & l_8 \\ -m_1 & n_7 & -n_8 \end{pmatrix} \begin{pmatrix} j_3 & l_8 & l_9 \\ -m_3 & n_8 & -n_9 \end{pmatrix} \begin{pmatrix} j_5 & l_9 & l_{10} \\ -m_5 & n_9 & -n_{10} \end{pmatrix} \times$$
$$\times \begin{pmatrix} j_4 & l_{10} & l_1 \\ -m_4 & n_{10} & -n_1 \end{pmatrix} \quad \text{(A.7.3)}$$

A.8 Some integrals in representation theory

We now list some integrals of matrix elements of SU (2) representations, used in the previous chapters to derive the generating functionals for discretized BF-theories and Group Field Theories.

Integral of the product of two matrix elements.

$$\int_G dg\ D^j_{m,n}(g)\ \left(D^{j'}_{m',n'}(g)\right)^* = \frac{1}{\dim(\rho)}\ \delta^{jj'}\ \delta_{mm'}\ \delta_{nn'}$$

Integral of the product of three matrix elements.

$$\int_G dh\ D^{j_1}_{n_1,k_1}(h)\ D^{j_2}_{n_2,k_2}(h)\ D^{j_3}_{n_3,k_3}(h) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\ \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$$

Integral of the product of four matrix elements.

$$\begin{aligned}\int_G dh\ \ D^{j_1}_{n_1,k_1}(h)\ D^{j_2}_{n_2,k_2}(h)\ D^{j_3}_{n_3,k_3}(h)\ D^{j_4}_{n_4,k_4}(h) = &\sum_{\substack{j,m,m'\\ n,n'}} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \times \\ \times\ &\begin{pmatrix} j & j_3 & j_4 \\ m' & m_3 & m_4 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j \\ n_1 & n_2 & n \end{pmatrix} \begin{pmatrix} j & j_3 & j_4 \\ n' & n_3 & n_4 \end{pmatrix} \\ \times\ &(-1)^{2j+m+n}\ (2j+1)\ \delta_{m,-m'}\ \delta_{n,-n'}\end{aligned}$$

Conclusions

The Group Field Theory approach to BF-theories provides a standard tool to derive actions and generating functionals for D-dimensional quantum models, using the well-known machineries of quantum field theory. By using this formalism, the observables one has to look for are, as a consequence, the propagators, the vertex functions and, in general, the scattering amplitudes between "states" (in this context regarded as geometries of the space-time), calculated by using the scattering matrix operator derived in quantum field theory. After that, one ought to ask for the renormalizability of this theory, by studying the equations of the renormalization group, i.e. the corrections for "dressed" propagators, the behaviour of the running coupling constant, the fixed points, and, in general, the scaling of the observables with respect to the scaling of momenta and energies. This problems are still an open question.
The main difficulty to be solved in Group Field Theory is that in the four-dimensional case it fails to describe General Relativity: this is because there is no way to impose the simplicity constraints on the field (at the level of group action) in order to reduce the generating functional one obtains to the generating functional for gravity. On the contrary, this is achieved in the framework of spin networks models by the Barrett-Crane and EPRL/FK models, which, for this reason, are considered as the better approximation for discrete quantum gravity in this context, at least in the four-dimensional case. BC and EPRL/FK models easily follow from the theory of spin foams, which of course is not free of problems; in fact it is not clear in that case how to construct a well-defined space of quantum states, invariant under the action of the group of diffeomorphisms. This is because there are some mathematical details to be solved: for instance, we need to introduce a well-defined measure which ensures the space of quantum states to be turned into a Hilbert space, and, if possible, to require this Hilbert space to be a separable one. Other mathematical difficulties arise for example in the calculations of path integrals, where we never have a well-defined class of measures and integrable functions according to those measures, apart from the case of the point particle in quantum mechanics.

In my opinion the most difficult problem arising in discrete quantum gravity is the coupling with matter fields. At the classical level, this can be achieved

with the introduction of spinor bundles to describe the gauge theory; as a consequence, the gauge group and the theory of connections may be more difficult to be treated, from a purely mathematical point of view. But the key point is that the different structure of the gauge theory reflects into the construction of the quantum physical states and Hilbert spaces, which now should not be only invariant under the action of the group of diffeomorphisms and the gauge group SU (2), but under the action of a more complicated algebraic structure. As a consequence, we will not have anymore the spin foams as described in this thesis, and the discrete path integral quantization cannot be introduced only with requirements on the representations of the group SU (2).
The coupling with matter is indeed essential if one wants to get some experimental proofs; in fact all the experimental informations we have in quantum field theories come from the scattering of particles, rather than from the self-interactions of the field with itself and the vertex corrections, which are the "observables" of a no-matter theory. Thus, coupling with matter fields is a necessary requirement, in order to obtain a sort of experimental proofs of this theory.

Another problem to be carried out is how to achieve the classical and the continuum limits. By means of the description introduced in this paper, it is not clear how the classical limit can be arranged, in terms of quantities, amplitudes and constants approaching a particular value. This is because quantum field theory has a natural "classical" limit in quantum mechanics, but, in our case, we do not have a quantum mechanics involving General Relativity. Because of this reason, the procedure of taking the limit is not well-defined, that is, we do not know which quantities in the equations makes the theory "quantum" instead of classical.
The argument of the continuum limit is also worth to be introduced. We have seen that the set of quantum states is made of the spin networks, which are combinatorical objects constructed out of a regolarization on the space-time. We are forced to think that, at some scales, the space-time manifold and the quantum states appear as a continuum, because of General Relativity; so it must exist a length, say L_0, such that, for all measurements done with parameters $l \geq L_0$, the set of quantum states must appear as continuous, i.e. we want the spin networks to appear as "glued together", so that they would become *indistinguishable* from a continuum of points. Neither in this case it is clear how to achieve this limit.

Apart from these considerations, we want to stress once more how a quantum field theory for the gravitational field can be introduced, i.e. the Group Field Theory. Even if there are a lot of mathematical details to be arranged, this framework is the natural point of arrival of the lattice gravity described in terms of spin foams, and this should be the starting point to include the coupling with matter fields into the gravitational interaction. The final remark deals with the flow of the renormalization group, which one has to derive in order to be sure that the theory is really renormalizable. The study of the equations of the flow of the running coupling constant, the asymptotic freedom and all other devices provided by the tools of quantum field theory are needed in order to give the physical understanding of the quantum theory of gravity.

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