Noncommutative Geometry and Particle Physics

Exercises

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Ex1 Find the states (and the pure ones) for the algebra of $\boxed{n \times n}$ matrices.

Ex2 Prove that $\boxed{\mathcal{N}_\phi}$ is an ideal.

Ex3 Perform the GNS construction for $\boxed{\mathrm{Mat}(n, \mathbb{C})}$ starting from a pure state.

Ex4 Given the algebra $\boxed{\mathcal{C}_0(\mathbb{R})}$ consider the two states $\boxed{\delta_{x_0}(a) = a(x_0)}$ and $\boxed{\phi(a) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2} a(x)}$. Find the Hilbert space in the two cases.

Ex5 Make $\boxed{\mathcal{A}^N}$ into a Hilbert module, and discuss its automorphisms.

Ex6 The algebras $\boxed{\mathbb{C}, \mathrm{Mat}(\mathbb{C}, n)}$ and $\boxed{\mathcal{K}}$, compact operators on a Hilbert space are all Morita equivalent. Find the respective bimodules.

Ex7 Take $\boxed{\mathcal{A} = \mathcal{C}(\mathbb{R})}$ and $\boxed{D = i\partial_x}$. Prove that the distance among pure states gives the usual distance among points of the line $\boxed{d(x_1, x_2) = |x_1 - x_2|.}$

Ex8 Find at least two good reasons for which the construction of last lecture cannot be performed (without changes) for a noncompact and Minkowkian spacetime.

Ex9 Consider the toy model for which $\boxed{\mathcal{A}}$ is $\boxed{\mathbb{C}^2}$ represented as diagonal matrices on $\boxed{\mathcal{H} = \mathbb{C}^m \oplus \mathbb{C}^n}$, and $\boxed{D = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix}}$. Find $\boxed{D_A}$.

Ex10 Consider the case I for which $\boxed{\mathcal{A}}$ is the algebra of two copies of function on a manifold, $\boxed{C_0(M) \times \mathbb{Z}_2 = C_0(M) \oplus C_0(M)}$, again represented as diagonal matrices on $\boxed{\mathcal{H} = L^2(M) \oplus L^2(M)}$, and $\boxed{D = i\partial\!\!\!/ \otimes \mathbb{1} + \gamma^5 \oplus D_F}$.

Ex11 Perform a similar construction for a Grand Unified Theory.

Ex Find the states (and the pure ones) for the algebra of $\boxed{n \times n}$ matrices.

We know the answer from quantum mechanics. Pure states correspond vectors of the Hilbert space on which the matrices act, i.e. $\boxed{n}$ dimensional vectors. Nonpure states to density matrices, i.e. hermitean matrices of trace $\boxed{1}$ and positive eigenvalues.

The first statement can be seen as follows. The algebra of matrices is also a Hilbert space with inner product $\boxed{\mathrm{Tr}\, b^* a}$. By Riesz theorem then any linear functional will be an element of the algebra. Then we set

$$\boxed{\phi(a) = \mathrm{Tr}\, \rho a}$$

$$\boxed{\phi(\mathbb{1}) = 1 \Rightarrow \mathrm{Tr}\, \rho = 1}$$

The fact that the matrix can be positive means that it should be Hermitean and without loss of generality we can choose a basis in which it is diagonal, and being positive with positive eigenvalues. The only way for this diagonal matrix not be be expressible as convex sum of other matrices of this kind is is to have all eigenvalues vanishing except one (which should have value one). This is a pure state, but, considering for example

$$\boxed{\rho = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}}$$

then defining $\boxed{\varphi = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}}$ we have $\boxed{\phi(a) = \operatorname{Tr} \rho a = \varphi^\dagger a \varphi}$

back

Ex Prove that $\boxed{\mathcal{N}_\phi}$ is an ideal

This is a one-line proof: follows from

$$\boxed{\phi(a^* b^* b a) \leq \|b\|^2 \phi(a^* a)}$$

since $\boxed{\|\phi\| = \sup\{|\phi(a)| \mid \|a\| \leq 1\} = 1}$, i.e. the norm, which is one, is the supremum over the vectors of norm less than one, and $\boxed{\|ab\| \leq \|a\| \, \|b\|}$

back

Ex Perform the GNS construction for $\boxed{\text{Mat}(n, \mathbb{C})}$ starting from a pure state.

We will do the construction in gory detail for the two dimensional case, the generalization being straightforward.

Consider the matrix algebra $\text{Mat}(n, \mathbb{C})$ with the two pure states

$$\phi_1 \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = a_{11} \ , \qquad \phi_2 \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = a_{22} \ .$$

The ideals of elements of 'vanishing norm' of the states ϕ_1, ϕ_2 are, respectively,

$$\mathcal{N}_1 = \left\{ \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix} \right\} \ , \qquad \mathcal{N}_2 = \left\{ \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} \right\} \ .$$

The associated Hilbert spaces are then found to be

$$\mathcal{H}_1 = \left\{ \begin{bmatrix} x_1 & 0 \\ x_2 & 0 \end{bmatrix} \right\} \;\simeq\; \mathbb{C}^2 = \left\{ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} \;,\; \left\langle X \right| X' \big\rangle = x_1^* x_1' + x_2^* x_2' \;.$$

$$\mathcal{H}_2 = \left\{ \begin{bmatrix} 0 & y_1 \\ 0 & y_2 \end{bmatrix} \right\} \;\simeq\; \mathbb{C}^2 = \left\{ X = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\} \;,\; \left\langle Y \right| Y' \big\rangle = y_1^* y_1' + y_2^* y_2' \;.$$

As for the action of an element $A \in \mathbf{M}_2(\mathbb{C})$ on $\mathcal{H}_1$ and $\mathcal{H}_2$, we get

$$\pi_1(A) \begin{bmatrix} x_1 & 0 \\ x_2 & 0 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 & 0 \\ a_{21}x_1 + a_{22}x_2 & 0 \end{bmatrix} \equiv A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;,$$

$$\pi_2(A) \begin{bmatrix} 0 & y_1 \\ 0 & y_2 \end{bmatrix} = \begin{bmatrix} 0 & a_{11}y_1 + a_{12}y_2 \\ 0 & a_{21}y_1 + a_{22}y_2 \end{bmatrix} \equiv A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \;.$$

The equivalence of the two representations is provided by the off-diagonal matrix

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ,$$

which interchanges 1 and 2 : $U\xi_1 = \xi_2$. Using the fact that for an irreducible representation any nonvanishing vector is cyclic, from (1) we see that the two representations can be identified.

The procedure generalize to arbitrary n, and also, with little work, to compact operators.

back

Ex Given the algebra $\boxed{\mathcal{C}_0(\mathbb{R})}$ consider the two states $\boxed{\delta_{x_0}(a) = a(x_0)}$ and $\boxed{\phi(a) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2} a(x)}$. Find the Hilbert space in the two cases.

In the first case $\mathcal{N}_\delta$ is composed of all functions which vanish at x_0. Two function belong to the same equivalence class of $\mathcal{C}_0(\mathbb{R})/\mathcal{N}_\delta$ if they differ by any function which vanishes at x_0. Therefore they must have the same value in x_0, and we can identify the class of equivalence with this value. Therefore the quotient is simply $\mathbb{C}$. Note that by the same token the vaue of the function at a point is also an irreducible representation of the algebra. Since the algebra is commutative the only IRR are indeed $\mathbb{C}$.

In the second case $\mathcal{N} = \emptyset$ since $\phi(a^*a) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2} |a(x)|^2$ cannot be zero if $a \neq 0$. We then have a faithful (but reducible) representation.

This state is clearly not pure, for example:

$$\phi(a) = \frac{1}{2}\phi_1(a) + \frac{1}{2}\phi_2(a) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} dx\, e^{-x^2} a(x) + \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} dx\, e^{-x^2} a(x)$$

back

Ex Make $\boxed{\mathcal{A}^N}$ into a Hilbert module, and discuss its automorphisms

This is almost trivial. An element of $A \in \mathcal{A}_N$ is just an N-ple of element of the algebra $\{A_i\}$ and I can define a right (left) module as

$$(aA)_i = aA_i \ ; \ (Aa)_i = A_i a$$

The inner product can be easily defined

$$\langle a, b\rangle_{\mathcal{A}} = \sum_i a_i^\dagger b_i$$

The corresponding norm is

$$\|(a_1, \cdots, a_N)\|_{\mathcal{A}} := \left\| \sum_{k=1}^{n} a_k^* a_k \right\| \ .$$

That $\mathcal{A}^N$ is complete in this norm is a consequence of the completeness with respect to its norm. Parallel to the situation of the previous example, when the algebra is unital, one finds that $\mathrm{End}_{\mathcal{A}}(\mathcal{A}^N) \simeq \mathrm{End}^0_{\mathcal{A}}(\mathcal{A}^N) \simeq \mathbb{M}_n(\mathcal{A})$, acting on the left on $\mathcal{A}^N$. The isometric isomorphism $\mathrm{End}^0_{\mathcal{A}}(\mathcal{A}^N) \simeq \mathbb{M}_n(\mathcal{A})$ is now given by

$$\mathrm{End}^0_{\mathcal{A}}(\mathcal{A}) \ni |(a_1, \cdots, a_N)\rangle \langle (b_1, \cdots, b_N)| \;\; \mapsto \;\; \begin{pmatrix} a_1 b_1^* & \cdots & a_1 b_N^* \\ \vdots & & \vdots \\ a_N b_1^* & \cdots & a_N b_N^* \end{pmatrix} \;,$$
$$\forall \; a_k, b_k \in \mathcal{A} \;,$$

which is then extended by linearity.

The automorphisms of this module are simple $N \times N$ matrices with entries in the element of the algebra. If $\mathcal{A}$ is the algebra Note that for $\mathcal{A} = \mathrm{Mat}(n, \mathbb{C})$ the same space is also a Hilbert *space*, i.e. a Hilbert module over $\mathbb{C}$

$$\langle a, b \rangle_{\mathbb{C}} = \sum_i \mathsf{Tr}\, a_i^\dagger b_i$$

While for $\mathcal{A} = C_0(M)$

$$\langle a, b \rangle_{\mathbb{C}} = \sum_i \int \mathsf{d}\mu\, a_i^\dagger b_i$$

and the completion of the integral gives L^2.

back

Ex The algebras $\boxed{\mathbb{C}, \mathrm{Mat}(\mathbb{C}, n)}$ and and $\boxed{\mathcal{K}}$, compact operators on a Hilbert space are all Morita equivalent. Find the respective bimodules

For any integer n the algebras $\mathbb{M}_n(\mathbb{C})$ and $\mathbb{C}$ are Morita equivalent and the equivalence $\mathbb{M}_n(\mathbb{C})$-$\mathbb{C}$ bimodule is just $\mathcal{E} = \mathbb{C}^n$. The left action of $\mathbb{M}_n(\mathbb{C})$ on $\mathcal{E}$ is the usual matrix action on a vector, while $\mathbb{C}$ acts on the right on each component of the vector. The hermitian products for any two vectors $u = (u_i)$ and $v = (v_i)$ are $\langle v|\, w\rangle_{\mathbb{C}} := \sum_{i=1}^n \bar{v}_i w_i$ and ${}_{\mathbb{M}_n(\mathbb{C})}\langle u, v\rangle := |u\rangle\langle v|$ which reads $\bar{u}_i v_j$ in components. It is immediate to verify relation (**??**):

$$ {}_{\mathbb{M}_n(\mathbb{C})}\langle v, w\rangle\, u = |v\rangle\langle w|\, u = v\,\langle w|\, u\rangle_{\mathbb{C}} \ . $$

In components all of the above expressions are simply $\sum_j \bar{v}_i w_j u_j$. The module $\mathcal{E} = \mathbb{C}^n$ is free as a right $\mathbb{C}$-module while it is projective (of finite type) as a left $\mathbb{M}_n(\mathbb{C})$-module and $\mathbb{C}^n = \mathbb{M}_n(\mathbb{C})p$ where $p = |v\rangle\langle v|$ is any rank-one projection.

Generalizing the procedure one shows that the algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a separable Hilbert space $\mathcal{H}$ is Morita equivalent to the algebra $\mathbb{C}$ with $\mathcal{H}$ the equivalence $\mathcal{K}(\mathcal{H})$-$\mathbb{C}$ bimodule and hermitian products ${}_{\mathcal{K}(\mathcal{H})}\langle v, w\rangle := |v\rangle\langle w|$ and $\langle v|\, w\rangle_{\mathbb{C}} := \langle v|\, w\rangle_{\mathcal{H}}$. Again $\mathcal{H} = \mathcal{K}(\mathcal{H})p$ where $p = |v\rangle\langle v|$ is any rank-one projection but now $\mathcal{H}$ is not finite generated (i.e. finite dimensional) over $\mathbb{C}$.

If M is a locally compact Hausdorff topological space, then from the previous considerations follows that for any integer n the algebra $\mathbb{M}_n(\mathbb{C}) \otimes C_0(M) \simeq \mathbb{M}_n(C_0(M))$ is Morita equivalent to the algebra $C_0(M)$. back

Ex Take $\boxed{\mathcal{A} = \mathcal{C}(\mathbb{R})}$ and $\boxed{D = i\partial_x}$. Prove that the distance among pure states gives the usual distance among points of the line $\boxed{d(x_1, x_2) = |x_1 - x_2|}$

Given a function $a(x)$ of the algebra we should consider the supremum of $|a(x_1) - a(x_2)$ subject to the condition on the norm of $[D, a] \leq 1$ is

$$\|[D, a]\| = \| \sup_x \|a'(x)\|$$

clearly the supremum is attained for any function which ha $|a'(x) = 1|$ in the interval $[x_1, x_2]$, and the behaviour of the function outside is irrelevant (it just just be such that it eventually goes to zero at infinity).

In more dimensions, with genric γ's with $\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}$ the calculation is more involved for g generic, it goes through a Lipschitz norm, and in the end one obtains the usual geodesic distance given by the metric tensor.

back

Ex Find at least two good reasons for which the construction of last lecture cannot be performed (without changes) for a noncompact and Minkowkian spacetime.

On a noncompact space a derivative operator such as the original Dirac operator does not have a discrete spectrum, and therefore our construction based on the eigenvalues of D will not count. This a technical infrared problem, and on can circumvent it putting the system “in a box”. A Minkowkian noncompact space in somewhat unnatural, and not compatible with causality, what happens to the light cones? We will see in the course of the lectures that something can be done to create some Minkwkian noncommutative geometry.

back

Ex Consider the toy model for which $\boxed{\mathcal{A}}$ is $\boxed{\mathbb{C}^2}$ represented as diagonal matrices on $\boxed{\mathcal{H} = \mathbb{C}^m \oplus \mathbb{C}^n}$, and $\boxed{D = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix}}$. Find $\boxed{D_A}$

The exercise really has to do with finding one forms. We will solve it using some more sophisticated mathematics, which will help us introduce some concepts we overlooked in the lecture.

Consider a space made of two points $Y = \{1, 2\}$. The algebra $\mathcal{A}$ of continuous functions is the direct sum $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ and any element $f \in \mathcal{A}$ is a pair of complex numbers (f_1, f_2), with $f_i = f(i)$ the value of f at the point i. An even spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is constructed as follows. The finite dimensional Hilbert space $\mathcal{H}$ is a direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and elements of $\mathcal{A}$

act as diagonal matrices

$$\mathcal{A} \ni f \mapsto \begin{pmatrix} f_1 \mathbb{1}_{\dim \mathcal{H}_1} & 0 \\ 0 & f_2 \mathbb{1}_{\dim \mathcal{H}_2} \end{pmatrix}$$

There is a natural grading operator γ given by

$$\gamma = \begin{pmatrix} \mathbb{1}_{\dim \mathcal{H}_1} & 0 \\ 0 & -\mathbb{1}_{\dim \mathcal{H}_2} \end{pmatrix} \ .$$

An operator D – being required to anticommute with γ – must be an off-diagonal matrix,

$$D = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix} \ , \quad M \in \mathrm{Lin}(\mathcal{H}_2, \mathcal{H}_1) \ .$$

With $f \in \mathcal{A}$, one finds for the commutator

$$[D, f] = (f_2 - f_1) \begin{pmatrix} 0 & M \\ -M^* & 0 \end{pmatrix} \ ,$$

and for its norm, $\|[D,f]\| = |f_2 - f_1|\lambda$ with λ the largest eigenvalue of the matrix $|M| = \sqrt{M^*M}$. Therefore, the noncommutative distance between the two points of the space is found to be

$$d(1,2) = \sup\{|f_2 - f_1| \mid \|[D,f]\| \leq 1\} = \frac{1}{\lambda} \ .$$

Since D is a finite hermitian matrix, this geometry is just '0-dimensional' and the only available trace is ordinary matrix trace.

It turns out that for this space, and for more general discrete spaces as well, it is not possible to introduce a real structure which fulfills all the requirements. It seems that it is not possible to satisfy the first order condition.

We now construct the exterior algebra on this two-point space. The space $\Omega^1\mathcal{A}$ of universal 1-forms can be identified with the space of functions on $Y \times Y$ which vanish on the diagonal.

Since the complement of the diagonal in $Y \times Y$ is made of two points, namely the pairs $(1,2)$ and $(2,1)$, the space $\Omega^1\mathcal{A}$ is two-dimensional and a basis is constructed as follows. Consider the function e defined by $e(1) = 1, e(2) = 0$; clearly, $(1-e)(1) = 0$, $(1-e)(2) = 1$. A possible basis for the 1-forms is then given by*

$$e\delta e \ , \quad (1-e)\delta(1-e) \ ,$$

and their values are

$$\begin{aligned} (e\delta e)(1,2) = -1 \ , &\quad ((1-e)\delta(1-e))(1,2) = 0 \\ (e\delta e)(2,1) = 0 \ , &\quad ((1-e)\delta(1-e))(2,1) = -1 \ . \end{aligned}$$

where I defined $\delta e \equiv [D, e]$.

*Here I am oversimplifying a construction based on cyclic cohomolgy.

Any 1-form can be written as $\alpha = \lambda e\delta e + \mu(1-e)\delta(1-e)$, with $\lambda, \mu \in \mathbb{C}$. One immediately finds that

$$e[D,e] = \begin{pmatrix} 0 & -M \\ 0 & 0 \end{pmatrix} \ , \tag{1}$$

$$(1-e)[D,1-e] = \begin{pmatrix} 0 & 0 \\ -M^* & 0 \end{pmatrix} \ , \tag{2}$$

and a generic 1-form $\alpha = \lambda e\delta e + \mu(1-e)\delta(1-e)$ is

$$\alpha = -\begin{pmatrix} 0 & \lambda M \\ \mu M^* & 0 \end{pmatrix} \ .$$

Hermitean one forms have $\mu = \bar{\lambda}$.

back

Ex Consider the case I for which $\boxed{\mathcal{A}}$ is the algebra of two copies of function on a manifold, $\boxed{C_0(M) \times \mathbb{Z}_2 = C_0(M) \oplus C_0(M)}$, again represented as diagonal matrices on $\boxed{\mathcal{H} = L^2(M) \oplus L^2(M)}$, and $\boxed{D = i\partial\!\!\!/ \otimes \mathbb{1} + \gamma^5 \oplus D_F}$.

The geometry in this case is almost commutative, i.e. the product of the ordinary commutative manifold times a finite dimensional space composed by two points. Note that the geometry is still commutative, the topological space is comprised of two copies of M. Accordingly with the considerations of the previous example γ and D_f are like the earlier D before, with the matrix M being just a complex number.

One forms split int two as well. The one due to $\partial\!\!\!/$ gives an Hermitean element of the algebra, which corresponds to the potential of a $U(1)\times U(1)$ theory, since the two elements on the diagonal matrix are different.

The part of the potential coming from D_F is instead a *scalar field* which connects the two sheets.

$$D_A = (D + \not{A}) \otimes \mathbb{1} + \gamma^5 \otimes \begin{pmatrix} 0 & \phi \\ \phi & 0 \end{pmatrix}$$

Note that the algebra is made of two diagonal elements, which are eigenvalues of γ with ± 1 eigenvalues, hence, heuristically speaking, they are "right" and "left" sheet. The field ϕ connect the two sheets, which is what the Higgs field actually does. This analogy becomes more precise with the spectral action.

back

Ex Perform a similar construction for a Grand Unified Theory.

The construction will not work for the majority of GUT's. The reason is that an algebra has less irreducible representations than a group.

The reason for this is simple, an algebra has to accommodate two operations, sum and product, while a group only has one.

The relevant case is Mat$(n, \mathbb{C})$. As algebra it has only one non trivial irreducible representation, the definitory one, i.e. $n \times n$ complex matrices. The unitary elements of the algebra form always a group, and in this case the group is $U(n)$, which has is known has an infinity of irreducible representations. If we take a m dimensional representation of $U(m)$ and start allowing also the matrices which we obtain using the sum besides the product then we unavoidably obtain Mat$(m, \mathbb{C})$ or Mat$(m, \mathbb{R})$ according to the representation, but not Mat$(n, \mathbb{C})$.

Partial exception are conplex numbars (one by one matrices) two by two matrices. For the former we have a representation for any real number, but the algebra one obtains is of course always $\mathbb{C}$. The two by two case is connected with the peculiar characteristics of Pauli matrices. In this case we have the possibility of using quaternions, whose unitary elements are $SU(2)$.

The construction with a gauge group (also made of different factors) requires fermions to transform either in the fundamental representation of the gauge group (or each of its factors) or the trivial one. This is what happens for the standard model, where quarks tranform under the the fundamental representation of $SU(3)$ and leptons under the trivial one, left particles are doublets of $SU(2)$ and right particles are singlets (transform under the trivial representation).

GUT's like $SU(5)$ require the fermions to be represented in a five dimensional spae (and this is OK) as well as 10 dimensional space, and this creates the problem. Likequse for $SO(10)$ which requires a 16 dimensional representation.

An exception is the Pati-Salam model based on $SU4) \times SU(2) \times SU(2)$. In this case lepton number becomes a "fourth colour" and right particles transform under the second $SU(2)$.

back