Do we know where are we?

Fedele Lizzi

Università di Napoli *Federico II*

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We know the answer to the question in the title of the talk:

Here.

By "here" we mean a particular point in space (and depending on the context, time).

The point of this talk is points.

In my elementary school book a **point** was defined as:

A Geometrical Entity without Dimension

I must confess that after reading it I was none the wiser about what a point is

Probably because I was convinced I knew what a point is. I could produce them at will with my biro. Or better with a sharper pin, or better. . .

Things are not so simple. . .

Euclid defined a point as **That which has no part**

The highest authority I can think of, *Wikipedia*, states:

In Euclidean geometry, a point is a primitive notion upon which the geometry is built.

Which fits well with what I was thinking in first grade, but it is not satisfactory at all ages. . .

Let's get to the point.

Points are ubiquitous in both the common language, physics and mathematics, but are we sure we are always talking of the same thing?

We usually think of points as something elementary and therefore "small". In physics this may be misleading.

In astrophysics a point may be a galaxy, or even a cluster of galaxies.

Before relativity, points in space and points in time were completely different objects for physicists. Points of time have a different name (instants).

Relativity introduces spacetime. Points become events. Gravity curves spacetime. Even if elementary, events implicitly assume some inner structure.

In some sense in physics points are “the places were things are”, which of course does not mean anything, unless we give a mathematic structure.

In classical mechanics we deal with elementary structures, point particles, and describe the state of motion by giving position and velocity, a point in phase space.

The structure is not given to points, which remain elementary, but to the relations among them.

Here we enter the real of mathematics. Define familiar mathematical objects: topology, differential, symplectic forms, (co)tangents . . .

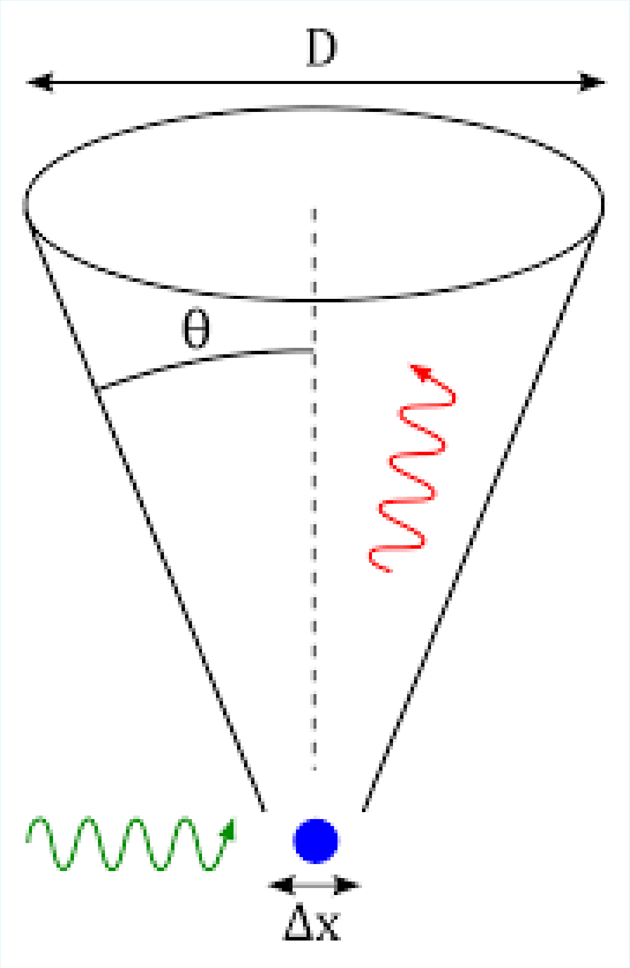
The notion of point is generalised, extended rigid bodies are described by a point in a higher dimensional space, fluids need infinite dimensions, but apart from mathematical difficulties, there are no problems in describing physical systems as points, and dynamics is their evolution in time.

Everything changed with **Quantum Mechanics**.

The first example of a Noncommutative Geometry.

Heisenberg uncertainty principle disposed with the idea of infinitesimal point in phase space. We cannot know with absolute precision at the same time position and velocity of a particle.

A very heuristic explanation is given by the so called Heisenberg Microscope.



Position and momentum, become noncommuting operators on a Hilbert space. The closest you may get to the concept of points are coherent states.

The physics of quantum phase space for nonrelativistic point particles is relatively well known. The founding fathers Bohr, Heisenberg, Dirac, Pauli . . . started the understanding of physics, and Weyl, Hilbert, Von Neumann and other gave the necessary mathematical framework.

I dare not affirm that quantum mechanics is well understood, but I venture to say that we have a solid mathematical framework with which we can work.

So far we had used only two of the fundamental constants: speed of light $\boxed{c}$, and Planck's constant $\boxed{\hbar}$.

If we attempt to define points in space(time) at very short distance we run into trouble if we put together quantum mechanics and gravity.

There is a phenomenon noticed for the first time by Bronstein in 1938, but presented independently in a modern and most terse way by Doplicher, Fredenhagen and Roberts in 1994.

I will present a caricature of these arguments, which however captures the main idea in a nontechnical way.

It is a variant of the Heisenberg microscope described above.

We are interested only is space, and not momentum, for which there is no limitation in quantum mechanics to an arbitrary precise measurement of $\boxed{x}$

Including gravity there is a length scale obtained combining the $\boxed{c}$, $\boxed{\hbar}$ and Newton's constant: $\boxed{\ell = \sqrt{\frac{\hbar G}{c^3}} \simeq 10^{-33}\text{cm}}$

In order to "measure" the position of an object, and hence the "point" in space, one has to use a very small probe, which has to be very energetic, but on the other side general relativity tells us that, if too much energy is concentrated in a small region, a black hole is formed.

It is possibly (ideally) to detect the BH, but not to "see" anything inside its horizon. Again there is a limit to the precision of the measurement.

For a rigorous statement we would need a full theory of **quantum gravity**. A theory which do not (hopefully yet) posses.

This is but an example of quantum spaces, loop quantum gravity is another. These spaces have in general in common the fact that the concept of point, of localisation, ceases to have a meaning below a certain scale.

Here we come to the question I asked in the title: Do we know where we are?

I will not attempt to define knowledge, not in front of this audience. . .

I will have instead a "tempered operationalist" position, as we did in my paper in Sinthese with Huggett and Menon, which I nearly verbatim quote in the following.

A necessary condition on a concept having physical content, and therefore to be a subject of knowledge, is that it is possible, by the lights of physical theory, to describe a (perhaps idealised) measurement procedure for a magnitude associated with the concept.

In a sense, inverting Ryle, we need a “machine in the ghost” to be able to elect a concept to the rank of being knowable.

We refer to such concepts as operationally definable. To give operationalism substance, one has to specify what measuring operations are available; since we are interested in the possibility of operationalizing points of space.

We need to be probabilistic, this is an absolute necessity if we consider a quantized theory, but also classically, every measurement is a probabilistic procedure

This is because any measuring device is affected by implicit, technological, limitations, and therefore every measure has an error.

The difference classical/quantum is the fact that in quantum spaces the uncertainty is inherent in the theory.

I note, in passing the importance of the presence of a dimensionful, physical scale.

In this context a measuring device is a mathematical entity, idealizing an actual physical instrument.

I will make an illustrative example in one dimension.

I will use a few formulas, but will illustrate with pictures.

I will approximate the classical state, a particle in a point with a Gaussian state, centered in a point $\boxed{x_0}$.

$$\boxed{\rho = \frac{1}{\sqrt{\pi}\alpha}\ e^{-\frac{(x-x_0)^2}{\alpha^2}}}$$

When $\boxed{\alpha}$ is small the state is well localised.

I will describe the measuring device, centred in $\boxed{x_1}$, with a Gaussian as well, with different normalisations:

$$\boxed{\frac{1}{\sqrt{\pi}\beta} g_1(x) = e^{-\frac{(x-x_1)^2}{\beta^2}}}$$

The probability of a measurement is proportional to (for simplicity set $\boxed{\alpha = \beta}$):

$$\boxed{e^{-\frac{(x_0-x_1)^2}{\alpha^2}}}$$

This quantity goes quickly to zero as the distance between $\boxed{x_0}$ and $\boxed{x_1}$ increases.

The decrease is more dramatic when $\boxed{\alpha}$ is big.

The interpretation is that our device measures practically zero, unless it fits exactly the position of the state.

Even if the framework is probabilistic, the parameter $\boxed{\alpha}$ can be arbitrarily large, and the state arbitrarily sharp.

Things change dramatically if we instead consider a quantum space. As an example I will consider a two dimensional space in which the two coordinates do not commute, i.e. if I multiply $\boxed{xy}$ I get a different result from $\boxed{yx}$. This is represented by the formula:

$$\boxed{[x, y] = \mathrm{i}\theta}$$

I have chosen this example because it is basically the basis of quantum mechanics, for which we have a very well developed formalism, and interpretations.

Different varieties of quantum spaces and noncommutative geometry, more realisitic, are considered. They are technically more cumbersome.

Indeed if you prefer you may iterpret what I will say in terms of phase space, with the two coordinates representing position and momentum.

The formalism needed is the theory of operators, states, Hilbert spaces.

Central to this formalism and its interpretation is the fact that all measurements are probabilistic

I stress the fact that the there is nothing misteriuous at this stage about the appearance of probabilities

Every measurement process (of a continuous quantity) is necessarily a probability density, also in classical physics.

But while in classical physics the limit is simply due to the precision of the devices, in quantum mechanics this is an actual limit.

And I stress again, this is possible (or due?) to he presence of dimensionful physical scales

I like to see the change of paradigm from the point of view of a noncommutative geometry.

I started this talk from the concept of point, and said it is at the basis of geometry.

It turns out that it is possible to equally well describe geometry in a dual way, starting from functions (a physicists would say fields) and their relations, which form an algebra.

This is a programme stated in the forties by Gelfand and Naimark, and vigorously impulsed by Alain Connes.

It turns out that the properties of usual spaces are encoded in the algebra of functions defined on them, functions can be represented as operators on Hilbert spaces, and other properties (metric, differential calculus . . .) are encoded by the action of other operators.

For example topology (relations among points) is encoded in the algebra of continuous functions, smoothness i that of differentiable ones, distances can be obtained with the use of differential operator and so on.

Points can be reconstructed as (pure) states of the algebra.

States are a well defined mathematical object, which associates a number to a en element of the algebra, subject to various conditions.

Pure states we need are simply the ones which evaluate the functions at a point. Non-pure (mixed) states are density probabilities, like the Gaussians earlier.

Once we have translated geometry into its algebraic counterpart, quantization consists in considering a noncommutative algebra, and repeat the analysis mutatis mutandis.

We need to retain some vestige of the classical state, which should be obtainable when the scale is very small.

This is can be obtained considering a deformation of the commutative algebra of functions. Introducing a new product so that the product depends on the order of the factors, fo example

$$\boxed{x \star y - y \star x = \mathrm{i}\theta}$$

Or more generally $\boxed{f(x,y) \star g(x,y) \neq g(x,y) \star f(x,y)}$

In such a way that when $\boxed{\theta \to 0}$ the algebra becomes the original, commutative one.

There is a new algebra, made of the same functions as before, and it is tempting to consider the presence of the underlying space.

But in attempting to repeat the analysis which lead us to find points from the algebra as states, we encounter the obstacle that the structure of states of this new algebra is different.

For example the simple product of two Gaussians as before, giving the probability to find a particle somewhere, made with the $\boxed{\star}$ product gives a non real oscillating function, without an obvious maximum.

There are other states, for example the equivalent of coherent states, which "resemble" localisation in a point, as long as we do not try to make them too sharply peaked.

Points are operationally definable because of Heisenberg's uncertainty!

I will not go deeper into technicalities, because I want to stimulate a discussion (fighting the incoming buffet) on the ontology of such a view of spacetime.

Points where our staring point. Speaking in what for me is foreign language, it can be said that they are part of the necessary ontology.

I stress again that they are not a symbolic device made to explain things, to reduce things to a know situation. Such as the ocean water waves are used to explain electromagnetic (or even probability) waves.

It is impossible to even start the discourse without points, even at an abstract formal level.

Yet, while we need points, we know that the novel paradigm necessitates their negation, in favour of other concepts.

To quote the paper with Huggett and Menon (who speak natively the foreign language above:

In theories of noncommutative space, we again assume - but this time for reductio - that there is an ontic state corresponding to an arbitrarily precisely localised particle. We construct the analogue of an epistemic state: a density operator. We then attempt to localise this epistemic state to an arbitrarily small area and discover that this leads to ascriptions of negative probabilities. Since these measures are not elements of the state space, this signals a pathology. The only way to avoid this pathology, we argue, is to drop the assumption that there is an ontic state corresponding to an arbitrarily precisely located particle. Thus, even in principle, it is not possible, to localise a particle below a certain area. Operationally, then, such areas - and a fortiori points - are undefinable.

To conclude

Do we know where are we?