Quantum Spacetime, Noncommutative Geometry and Observers

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One often starts a talk with thanking the organisers, and I am no exception in thanking TRG, Sanatan, Sunajay and the other organisers for making this conference happen. But I would like to add another acknowledgements in the form of a quote from my 1985 doctoral thesis:

"After five years in Syracuse there are many people I would like to thank. First, and foremost, my advisor Professor A.P. Balachandran, he taught me most of what I know, and reproached me for not studying most of what I don't, and should, know. He showed me how to work and gave me the honour to be one of his collaborators and friends. Thanks, Bal!"

In 2026, I were to write a "retirement thesis", I would not change a word, except that "five years in Syracuse" would become "forty-six years in Syracuse, Oxford, Trieste, Napoli and Barcelona". But the "first and foremost" I would keep.

Gravity is the theory of (curved) spacetime. Its dynamical variable is spacetime itself, and in fact one way to quantize it is to consider the metric $\boxed{g^{\mu\nu}}$ as field the to quantize.

This attempt has not been completely successful. And I hasten to add that it is in good company, no attempt has been completely successful. Otherwise this conference would have been very different...

The idea behind the Noncommutative Geometry approach to Quantum Gravity is that the object to quantize is spacetime itself, giving thus rise to a Quantum Spacetime. I will concentrate on kinematics, describing the space by a noncommutative algebra, which can be sometimes described by noncommuting coordinate functions.

The most famous noncommutative space is the one described by noncommuting coordinate, whose commutator is constant. Sometimes it is called DFR (Doplicher, Fredenhagen, Robers) noncommutativity, or even Moyal or Gronëwold-Moyal, who introduced the deformed product which generalises this kind of noncommutativity. It also featured in the famous article of Seiberg and Witten on noncommutative geometry and strings.

$$\boxed{[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu}}$$

This is a spacetime replica of the quantum phase space canonical commutation relations, with $\boxed{\hbar}$ substituted by $\boxed{\theta}$. This meant that we could use all the experience and technology acquired for quantum mechanics.

The unpalatable issue is that this kind of noncommutativity breaks Lorentz invariance, although it maintains translation invariance

This was not a problem for quantum mechanics since there is not, in general, a symmetry rotating coordinates into momenta. For spacetime Lorentz however transformations are a must.

The breaking of Lorentz symmetry implies the presence of a fundamental antisymmetric tensor, i.e. two directions, a vector and a pseudovector, which would characterize our universe.

These fundamental directions would show up in cosmology, and the recent data pose stringent limits on this.

Quantization of gravity cannot prescind from symmetries

We may say that a quantum spacetime will require quantum symmetries

Quantum Groups and Hopf Algebras developed in parallel to the one of Non-commutative Geometry, with several intersections.

Of particular interest for us is the deformation of the Poincaré Lie algebra which goes under the name of $\boxed{\kappa}$-Poincaré. I can only mention a few features of it, since I want to concentrate on the role of the observers. It suffices to say that the homogeneous space for this quantum symmetry is generated by the commutation rule

$$\boxed{[x^0, x^i] = \mathrm{i}\lambda x^i \ ; \ [x^i, x^j] = 0}$$

Later if there is time we will discuss a space which is a variation on this theme.

I will study this space using the usual techniques of quantum mechanics. Let me first briefly recall a well known case study: Quantum Phase Space of a particle.

Phase space is a six-dimensional space spanned by $\boxed{(q^i, p_i)}$. Quantization introduces the commutation relation $\boxed{[q^i, p_j] = \mathrm{i}\hbar\delta^i_j \ ,}$

The most common representations of position and momenta is operators on $\boxed{L^2(\mathbb{R}^3_q)}$

$$\boxed{\widehat{q}^i\psi(q) = q^i\psi(q) \ ; \qquad \widehat{p}_i\psi(q) = -\mathrm{i}\hbar\frac{\partial}{\partial q^i}\psi(q) \ .}$$

$\boxed{\widehat{q}}$'s and $\boxed{\widehat{p}}$'s are unbounded selfadjoint operators with a dense domain. The spectrum is the real line (for each $\boxed{i}$).

They have no eigenvectors but improper eigenfunctions: distributions.

Since the $\widehat{q}^i$'s commute it is possible to have a simultaneous improper eigenvector of all of them, these are the Dirac distributions $\boxed{\delta(q-\bar{q})}$ for a particular $\boxed{\bar{q}}$ vector in $\boxed{\mathbb{R}^3}$ For a particular momentum $\boxed{\bar{p}}$ the improper eigenfunctions of the $\boxed{\widehat{p}_i}$ are plane waves $\boxed{\mathrm{e}^{\mathrm{i}\bar{p}_i q^i}}$.

Formally, the eigenvalue equation $\boxed{\partial_q \psi(q) = \alpha\psi(q) \ , \alpha \in \mathbb{C}^3}$ is solved by $\boxed{\mathrm{e}^{\alpha \cdot q}}$ with a vector $\boxed{\alpha}$

No function of this kind is square integrable, there are no (proper) eigenfunctions. The operator $\boxed{\widehat{p}}$ is self-adjoint on the domain of absolutely continuous functions. $\boxed{\alpha}$ must be pure imaginary because the distributions must be well defined on the domain of selfadjointness of the operators.

The improper eigenfunctions of momentum are physically interpreted as infinite plane waves of precise frequency. Since plane waves are not vectors of the Hilbert space there is no quantum state which would give as measure exactly the value $\hbar k$, nevertheless we have all learned to live with this fact, and there is a well-defined sense in which we talk about "particles of momentum $\boxed{\hbar k}$".

Implicitly we have chosen $\boxed{\widehat{q}^i}$ as a complete set of observables, the description of a quantum state as a function of positions. $\boxed{|\psi(q)|^2}$ (normalized) is the density probability to find the particle at position $\boxed{q}$.

The $\boxed{\psi}$ is complex and contains also the information about the density probability of the momentum operator.

We could have chosen $\boxed{\widehat{p}}$ as complete set. Then we would have the Fourier transformed $\boxed{\phi(p)}$. It is important that the Fourier transform is an isometry, it maps normalized functions of positions into normalized functions of momenta.

And we have other choices for complete sets, number operators, angular momentum ...

A different observer will have his own Hilbert space, set of observables, and so on. Unavoidably we have a tensor product.

This is usually not a problem, we know that we can make a unitary transformation from one set to another. Implicit we have a coproduct, i.e. a way to put together representations.

Quantum Groups have taught us that there is more than just the Lie algebra structure, symmetries are described by Hopf Algebra, since I have to put together representations.

Usually this is done in a cocommutative way, and we do not notice it. But if the algebra is deformed, so is the group.

A Hopf Algebra has additional structures in additions to being an algebra, i.e. a set with two operations and some other properties, think for example at a Lie algebra or the set of functions on some manifold. The latter is associative, the former is not (but it has Jacobi).

While a product is a map from two elements of the algebra into a single one, the coproduct tells how to put together representations, formally it is a map from one copy of the algebra into the tensor product of two copies. Something like

$$\boxed{\Delta(f) = f \otimes 1 + 1 \otimes f}$$

which can be seen as a rendition of the Leibnitz rule if the Lie algebra is represented as differential operators or

$$\boxed{\Delta(f) = f \otimes f}$$

which instead is relevant for the case of functions on a group for which $\boxed{f(gg') = f(g)f(g')}$

Two more structures, counit and antipode, duals of the unity and the inverse are important but not relevant for this seminar.

Earlier we discusse quantum phase space, where the deformation parameter was $\boxed{\hbar}$.. I now want to reproduce this discussion for $\boxed{\kappa}$-Minkowski, four dimensional space with different commutation relation and a deformation parameter $\boxed{\lambda = \frac{1}{\kappa}}$

This is a quantum space, but I will only consider its kinematic, and leave $\boxed{\hbar}$ alone for the moment.

But is a relativistic space, hence we need will worry about Poincaré transformations.

Look for a representation of the $\boxed{x^\mu}$ on $\boxed{L^2(\mathbb{R}^3)}$:

$$\boxed{\widehat{x}^i \psi(x) = x^i \psi(x)}$$

$$\boxed{\widehat{x}^0 \psi(x) = \mathrm{i}\lambda \left(\sum_i x^i \partial_{x^i} + \frac{3}{2} \right) \psi(x) = \mathrm{i}\lambda \left(r\partial_r + \frac{3}{2} \right) \psi(x).}$$

Positions are multiplicative operators, time is dilation. The $\boxed{3/2}$ factor is necessary to make the operator symmetric. It is selfadjoint on all absolutely continuous functions.

For dilations the polar basis is appropriate. The commutation relations and uncertainty principle become:

$$\boxed{[\widehat{x}^0, \cos\theta] = [\widehat{x}^0, \mathrm{e}^{\mathrm{i}\varphi}] = 0 \ , [x^0, r] = \mathrm{i}\lambda r.}$$

$$\boxed{\Delta x^0 \Delta r \geq \frac{\lambda}{2} |\langle r \rangle|.}$$

What is the spectrum of the time operator? Monomials in r are formal solutions of the eigenvalue problem:

$$\boxed{\mathrm{i}\lambda\left(r\partial_r + \frac{3}{2}\right) r^{\alpha} = \mathrm{i}\lambda(\alpha + \frac{3}{2})r^{\alpha} = \lambda_{\alpha} r^{\alpha},}$$

The eigenvalues are real if and only if $\boxed{\alpha = -\frac{3}{2} + \tau}$ with $-\infty < \tau < \infty$ a real number.

For momentum we had plane waves, in this case we have the following distributions

$$\boxed{T_{\tau} = \frac{r^{-\frac{3}{2}-\mathrm{i}\tau}}{\lambda^{-\mathrm{i}\tau}} = r^{-\frac{3}{2}} \mathrm{e}^{-\mathrm{i}\tau \log(\frac{r}{\lambda})}}$$

The distribution has the correct dimension of a length$\boxed{^{3/2}}$, the factor of $\boxed{\lambda}$ is there to avoid taking the logarithm of a dimensional quantity. Since $\boxed{\lambda}$ is a natural scale for the model, its choice is natural, but not unique.

For quantum phase space we had as complete set of observables either three $\boxed{q}$ or three $\boxed{p}$, connected by a Fourier transform,

For $\boxed{\kappa}$-Minkowski we have either $\boxed{(r,\theta,\varphi)}$ or $\boxed{(\tau,\theta,\varphi)}$, and we switch among the two with a Mellin transform

$$\boxed{\psi(r,\theta,\varphi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\tau\, r^{-\frac{3}{2}} \mathrm{e}^{-\mathrm{i}\tau \log\left(\frac{r}{\lambda}\right)} \widetilde{\psi}(\tau,\theta,\varphi) = \mathcal{M}^{-1}\left[\widetilde{\psi}(\tau,\theta,\varphi),\ r\right],}$$

$$\boxed{\widetilde{\psi}(\tau,\theta,\varphi) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \mathrm{d}r\, r^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\tau \log\left(\frac{r}{\lambda}\right)} \psi(r,\theta,\varphi) = \mathcal{M}\left[\psi(r,\theta,\varphi), \frac{3}{2}+\mathrm{i}\tau\right].}$$

$\boxed{|\psi|^2}$ and $\boxed{|\widetilde{\psi}|^2}$ can be interpreted as the probabilty density to find the particle in position $\boxed{r}$ or time $\boxed{\tau}$ respectively

It is useful to have an idea of the dimensional quantities involved.

Call $\boxed{t}$ the eigenvalue of the time operator $\boxed{\frac{x^0}{c}}$, then $\boxed{\tau = t\frac{c}{\lambda}}$.

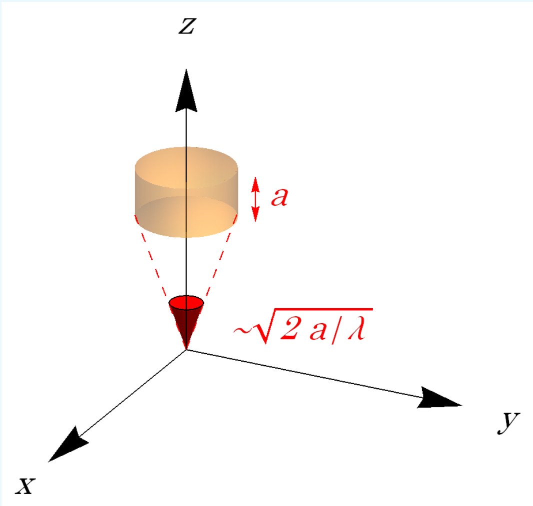
$\boxed{\frac{c}{\lambda}}$ is a dimensional quantity. If we choose for $\boxed{\lambda}$ the Planck length then $\boxed{\frac{c}{\lambda} \sim 2 \cdot 10^{43}}$ Hz. In other words if $\boxed{t = 1}$ s, then $\boxed{\tau = 2 \cdot 10^{43}}$, an extremely large number.

If $\boxed{t}$ is of the order of Planck time, then $\boxed{\tau \sim 1}$.

I will now give some examples of localised state, at the origin and away

Consider the following state (chosen to simplify calculations) localised in space in a small region of size $\boxed{a}$ around a point at distance $\boxed{z_0}$ along the $\boxed{z}$ axis.

$$\boxed{\psi_{z_0,a}(r,\theta,\varphi) = \begin{cases} \sqrt{\frac{3\lambda}{2a\pi((a+z_0)^3-z_0^3)}}, & z_0 \leq r \leq (z_0+a) \text{ and } \cos\theta > 1-\frac{a}{\lambda} \\ 0, & \text{otherwise} \end{cases}}$$



In the limit $\boxed{a \to 0}$ the state is localised in $\boxed{z_0}$

The Mellin transform of this function, integrating out the angular variables, gives:

$$\boxed{\int |\widetilde{\psi}_{z_0,a}|^2 \sin\theta \, \mathrm{d}\theta = \left[\frac{a}{4\pi^2 z_0} - \frac{a^2}{8\lambda(\pi^2 z_0^2)} + \mathcal{O}(a^3)\right]}$$

This tends to a constant which vanishes as $a \to 0$. Localising in space implies delocalising in time

The series expansion for $\boxed{a}$ around $\boxed{0}$, and $\boxed{z_0}$ around $\boxed{\infty}$, are the same. $\boxed{|\widetilde{\psi}_{z_0}|^2 = \frac{\lambda}{4\pi^2 z_0} - \frac{a\lambda}{8\pi^2 z_0^2} + \frac{a^2\lambda(7-4\tau^2)}{192\pi^2 z_0^3} + \mathrm{O}\left(a^3\right)}$

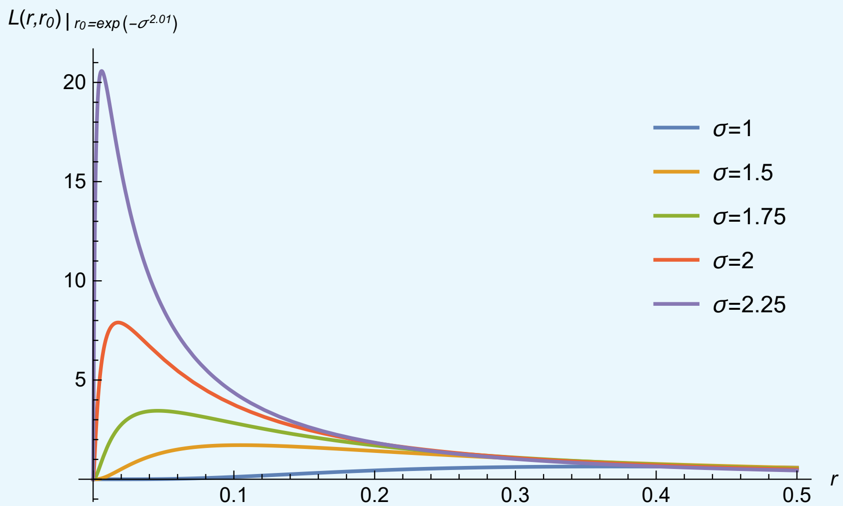
This means that a sharp localization of a particle far away from the origin implies that the particle cannot be localised in time. In accordance with the uncertainty for $\boxed{\kappa}$-Minkowski.

It is impossible to sharply localise a state at a point, except at the origin $\boxed{x^i = 0}$, which is an exceptional point.

The equivalent of the Gaussians of ordinary quantum mechanics are the log-Gaussians

$$\boxed{L(r, r_0) = N\mathrm{e}^{-\frac{(\log r - \log r_0)^2}{\sigma^2}} = \mathrm{e}^{-\left(\frac{\log\left(\frac{r}{r_0}\right)}{\sigma}\right)^2} \frac{\mathrm{e}^{-\frac{9}{16}\sigma^2}}{\sqrt{\sigma}(2\pi)^{3/4}\sqrt{r_0^3}}}$$

They have a maximum in $\boxed{r = r_0}$, which localises at $\boxed{r = r_0}$ as $\boxed{\sigma \to 0}$, and localises at $\boxed{r = 0}$ as $\boxed{r_0 \to 0}$, for any value of $\boxed{\sigma \geq 0}$.



Their Mellin transform are ordinary Gaussians (up to phases and normalizations) independent on $\boxed{r_0}$

$$\boxed{\widetilde{L}(\tau, r_0) = \frac{\sigma^{\frac{1}{2}} e^{-\frac{1}{4}\sigma^2 \tau(\tau - 3i)} r_0^{i\tau}}{2\sqrt[4]{2}\pi^{3/4}}}$$

In the double limit $\boxed{r_0 \to 0}$ and $\boxed{\sigma \to \infty}$, all $\boxed{\langle r^n \rangle_L}$ and all $\boxed{\langle (x^0)^n \rangle_L}$ go to zero as $\boxed{\sigma \to \infty}$.

This is a state localised both in space (at $\boxed{r = 0}$) and in time (at $\boxed{\tau = 0}$)

Localisation at arbitrary time is simply achieved multiplying the state by $\boxed{\left(\frac{r}{\lambda}\right)^{i\tau_0}}$

With the usual abuse of notation we will call these state as $\boxed{|o_\tau\rangle}$.

We have argued that the origin is a special point. Does this mean that somewhere in the universe there is "the origin". A special position in space singled out by the $\boxed{\kappa}$-God?

Implicitly in our discussion, when we were referring to states we were assuming the existence of an observer measuring the localisation of states.

This observer is located at the origin, and he can measure with absolute precision where she is. For him "here" and "now" make sense. She cannot localise with precision states away from him, as a consequence of the noncommutativity of $\boxed{\kappa}$-Minkowski.

What about other observers? A different observer will be in general Poincaré transformed, i.e. translated, rotated and boosted. These operations are usually performed with an element of the Poincaré group. But now we have $\boxed{\kappa}$-Poincaré!

Require invariance under the transformation $\boxed{x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1}$

But now the coordinate functions on the group are noncommutative, they are (in a particular basis, Zakrzewski)

$$\boxed{[a^\mu, a^\nu] = \mathrm{i}\lambda\,(\delta^\mu{}_0\, a^\nu - \delta^\nu{}_0\, a^\mu)\ , \qquad [\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma] = 0}$$

$$\boxed{[\Lambda^\mu{}_\nu, a^\rho] = \mathrm{i}\lambda\left[(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0)\,\Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu\right)\eta^{\mu\rho}\right].}$$

In particular notice that translations are now noncommuting. With the same commutation relations of the coordinates.

We represented the $\boxed{\kappa}$-Minkowski algebra as operators. But in doing so we had implicitly chosen an observer.

In order to take into account the fact that there are different observers we enlarge the algebra (and consequently the space) to include the parameters of the new observers. We call then new set of states as $\boxed{\mathcal{P}_\kappa}$

Our (generalised) Hilbert space will now comprise not only functions on space-time (either functions of $\boxed{r}$ or $\boxed{\tau}$), but also functions of the $\boxed{a}$'s and $\boxed{\Lambda}$'s.

We can represent the $\boxed{\kappa}$-Poincaré group faithfully as

$$\boxed{a^\rho = -\mathrm{i}\,\frac{\lambda}{2}\left[\left(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0\right)\Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu\right)\eta^{\mu\rho}\right]\Lambda^\nu{}_\alpha \frac{\partial}{\partial \omega^\mu{}_\alpha} + \mathrm{i}\frac{\lambda}{2}\left(\delta^\rho{}_0\, q^i \frac{\partial}{\partial q^i} + \delta^\mu{}_i\, q^i\right) + \frac{1}{2}\mathrm{h.c.}}$$

Where $\boxed{\omega}$ are the parameters of the Lorentz transformation, and the $\boxed{\Lambda}$'s are represented as multiplicative operators

We have therefore that, like spacetime, the space of observers is also noncommutative, and the noncommutativity is only present in the translation sector.

We now explore the space of observers, seen as states. First consider the observer located at the origin, which is reached via the identity transformation.

Define $\boxed{|o\rangle_{\mathcal{P}}}$ with the property:

$$\boxed{{}_{\mathcal{P}}\langle o|\, f(a,\Lambda)|o\rangle_{\mathcal{P}} = \varepsilon(f)\,,}$$

with $\boxed{f(a,\Lambda)}$ a generic noncommutative function of translations and Lorentz transformation matrices, and $\boxed{\varepsilon}$ the counit.

This state describes the Poincaré transformation between two coincident observers. The state is such that all combined uncertainties vanish. Coincident observers are therefore a well-defined concept in $\boxed{\kappa}$-Minkowski spacetime.

A change of observer will transform $\boxed{x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1}$ and primed and unprimed coordinates correspond to different observers.

Identifying $\boxed{x}$ with $\boxed{\mathbb{1} \otimes x}$ we generate an extended algebra $\boxed{\mathcal{P} \otimes \mathcal{M}}$ which extends $\boxed{\kappa}$-Minkowski by the $\boxed{\kappa}$-poincaré group algebra.

This algebra takes into account position states and observables

Remember that, just as we cannot sharply localise position states, neither we can sharply localise where the observer is.

Since Lorentz transformations commute among themselves, we can however say if two observers are just rotated with respect to each other

We can build the action of the position, translation and Lorentz transformations operator on generic functions of all those variables.

To simplify notations let us consider $\boxed{1+1}$ dimensions. In this case there are only two position coordinates, two translations coordinates and one Lorentz transformation parametrized by $\boxed{\xi}$

The relations are $\boxed{\Lambda^0{}_0 = \Lambda^1{}_1 = \cosh\xi\,,\ \Lambda^0{}_1 = \Lambda^1{}_0 = \sinh\xi\,,}$

$$\boxed{[a^0, a^1] = \mathrm{i}\lambda\, a^1\,, \qquad [\xi, a^0] = -\mathrm{i}\lambda \sinh\xi\,, \qquad [\xi, a^1] = \mathrm{i}\lambda\,(1-\cosh\xi)\ .}$$

And the action on $\boxed{\mathcal{P}}$ is

$$\boxed{a^0 = \mathrm{i}\lambda q\frac{\partial}{\partial q} + \mathrm{i}\lambda \sinh\xi\,\frac{\partial}{\partial \xi}\,,\ a^1 = q + \mathrm{i}\lambda\,(\cosh\xi - 1)\,\frac{\partial}{\partial\xi}\,,}$$

States (non entangled) will be objects of the kind $\boxed{|g\rangle \otimes |f\rangle}$

In particular $\boxed{|g\rangle \otimes |o\rangle}$ is a pure translation of the state at the origin.

The new observer measures coordinates with $\boxed{x'}$. The expectation values on (normalised) transformed state is

$$\boxed{\langle x'^{\mu}\rangle = \langle g| \otimes \langle o|x'^{\mu}|g\rangle \otimes |o\rangle = \langle g|\Lambda^{\mu}{}_{\nu}|g\rangle\langle o|x^{\nu}|o\rangle + \langle g|a^{\mu}|g\rangle\langle o|o\rangle\,,}$$

We get:

$$\boxed{\langle x'^{\mu}\rangle = \langle g|a^{\mu}|g\rangle\,,}$$

The expectation value of the transformed coordinates is completely defined by translations. This is natural, the different observers are comparing positions, not directions.

In general

$$\boxed{\langle x'^{\mu_1} \dots x'^{\mu_n} \rangle = \langle g | a^{\mu_1} \dots a^{\mu_n} | g \rangle \langle o | o \rangle = \langle g | a^{\mu_1} \dots a^{\mu_n} | g \rangle \ .}$$

Poincaré transforming the origin state $\boxed{|o\rangle}$ by a state with wave function $\boxed{|g\rangle}$ in the representation of the $\boxed{\kappa}$-Poincaré algebra, the resulting state will assign, to all polynomials in the transformed coordinates the same expectation value as what assigned by $\boxed{|g\rangle}$ to the corresponding polynomials in a^μ.

In other words, the state $\boxed{x'^\mu}$ is identical to the state of $\boxed{a^\mu}$.

All uncertainty in the transformed coodinates $\boxed{\Delta x'^\mu}$ is introduced by the uncertainty in the state of the translation operator, $\boxed{\Delta a^\mu}$.

It is also possible to see that the uncertainty of states increases with translation.

I can summarise saying that all observers can sharply localise states in their vicinity, and cannot localise states far away from them.

The apparent paradox of a state badly localisable by Alice, but which is well localised by Bob, is that Bob herself is badly localised by Alice, and of course viceversa.

All this is qualitatively perfectly compatible with the principle of relative locality (Amelino-Camelia, Kowalski-Glickman, Freidel, Smolin), which however starts in a quite different context: curved momentum space. In this analysis instead momentum does not appear explicitly, although it is present in the symmetry.

One of the tenets of Quantum Mechanics is that the observer is classical, usually macroscopic, and that therefore we “know” how to deal with them.

In quantum gravity this may not be the case. While it is true that the smallness of the Planckian constants suggests this, there may be amplifying effects, and conceptual aspects to deals with.

The group algebra approach, where the parameters of the Poincaré transformations do not commute is the key to understand the observer-dependent transformations

Transformations relating different frames belong to a noncommutative algebra. Hence localisability limitations.

Alternatively, the deformation can be seen as a deformation of the tensor product. This is evident in the case of a Drinfeld twist, and I give another example, based on a twist.

$\boxed{\varrho}$-Minkowski or Angular Noncommutativity

$$\boxed{[x^0, x^1] = -\mathrm{i}\varrho x^2 \ ; \ [x^0, x^2] = \mathrm{i}\varrho x^1 \ ; \ [x^0, x^3] = 0 \ ; \ [x^1, x^j] = 0}$$

This form of noncommutativity has a long history, Gutt, Lukierski, Woronowicz, Chaichian, Demichev, Presnajder, Tureanu and more recently Amelino-Camelia, Barcaroli, Loret, Bianco and Pensato.

A similar version can be built in which $\boxed{x^0}$ and $\boxed{x^3}$ are exchanged. I will not discuss this variant here.

Express the commutation relations in cylindrical coordinates $\boxed{(t, \rho, z, \varphi)}$

$$\boxed{\text{“}[t, \varphi] = \mathrm{i}\varrho\text{”}\,; \;\; [t, z] = [t, \rho] = \text{“}[\rho, \varphi]\text{”} = [\rho, z] = 0}$$

Note that I have put some of the commutators in inverted commas.

We can repeat the previous analysis, but take into account that the angular variables are not good observables. This explains the inverted commas.

A better expression would be $\boxed{[r, Y(\theta, \varphi)] = 0}$, where $\boxed{Y}$ is an operator generated by well defined functions of $\boxed{\theta, \varphi}$.

This time the uncertainty will be between time and the angular variable. And one should definitely resist the temptation to write:

$$\boxed{\cancel{\Delta t \Delta \varphi \geq \frac{\varrho}{2}}}$$

In the $\boxed{\{\rho, z, \varphi\}}$ basis $\boxed{t}$ is represented by the derivation operator $\boxed{-\mathrm{i}\varrho\partial_{\varphi}}$.

This operator has Discrete Spectrum!

A change of basis is given by the Fourier series. The eigenstates of momentum are $\boxed{\mathrm{e}^{\mathrm{i}n\varphi}}$, and they are completely delocalised in $\boxed{\varphi}$

On the other hand, a state completely localised in $\boxed{\varphi}$, given by a $\boxed{\delta}$, which requires a superposition with equal weights of all eivenvalues of time.

$$\boxed{\delta(\varphi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\varphi}}$$

After a time measurement, which has given as result $\boxed{n_0 \varrho}$, the system is in the eigenstate $\boxed{\mathrm{e}^{\mathrm{i}n_0\varphi}}$.

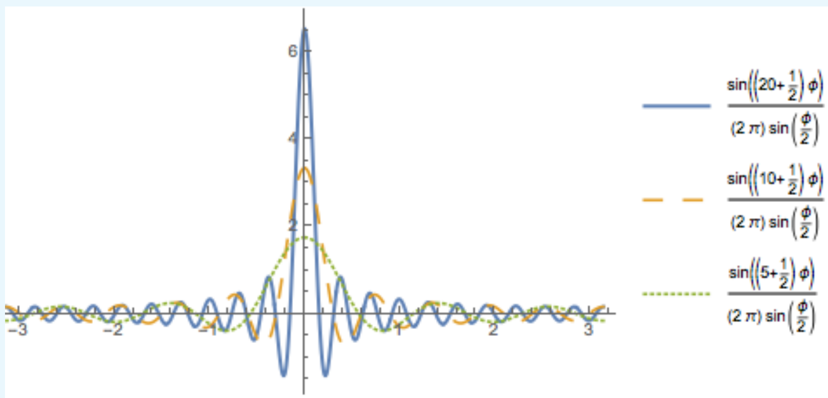
A slightly uncertain state uses a great number of Fourier modes to built a state peaked around some time, then the corresponding uncertainty is the angular variable is given by the fact that only a finite set of elements of the basis are available.

For $\boxed{\varrho}$ Planckian of the quantum of time (also called a chronon), is $\boxed{5.39\ 10^{-44}\, sec.}$

The most accurate measurement of time is $\sim 10^{-19}\, sec.$ Heuristically the superposition of 10^{35} quanta of time is needed.

Approximate δ by the Dirichlet nucleus $\delta_N = \sum_{n=-N}^{N} \mathrm{e}^{in\varphi} = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})\varphi}{\sin \frac{N}{2}\varphi}$

For $N = 5, 10, 15$.



The needs $N \sim 10^{35}$. Then the first zero of the nucleus is at $\varphi \sim 10^{-35}$. We may assume this to be the uncertainty in an angle determination. To translate this as an uncertainty in position we need ρ. For the radius observable universe ($10^{26}\, m$) the uncertainty is of the order of one metre.

Is this all pervading clicking a feature of our universe? Is time translation definitely lost? Putting time on a lattice may be disturbing.

Self-adjointness come to the rescue. Anybody who has studied the Aharonov-Bohm experiment knows that the momentum operator on a compact domain is a rich operator.

It is self-adjoint on periodic functions, but is also selfadjoint on functions periodic up to a phase. In this case the eigenfunctions are $\boxed{\mathrm{e}^{\mathrm{i}(n+\alpha)\varphi}}$.

The differences between states is unchanged, and the effect is a rigid shift. This however means that a different choices of selfadjointess domains. Time translations are undeformed, and two time translated observers will be in different, but equivalent domains.

In order to compare their results the two observers, again, have to compare representantions, and this is ruled by a coproduct.

Noticing that $\boxed{[\partial_t, \partial\varphi] = 0}$, the deformation can be built with a Drinfeld twist.

$$\boxed{\mathcal{F}(x,y) = \exp\left\{-\frac{\mathrm{i}\varrho}{2}\left(\partial_{y^0}\left(x^2\partial_{x^1} - x^1\partial_{x^2}\right) - \partial_{x^0}\left(y^2\partial_{y^1} - y^1\partial_{y^2}\right)\right)\right\}}$$

$$\boxed{= \exp\left\{\frac{\mathrm{i}\varrho}{2}\left(\partial_{y^0}\partial_{\varphi_x} - \partial_{x^0}\partial_{\varphi_y}\right)\right\}}$$

This deforms the Hopf algebra as

$$
\begin{aligned}
\Delta P_3 &= P_3 \otimes 1 + 1 \otimes P_3, \\
\Delta P_0 &= P_0 \otimes 1 + 1 \otimes P_0, \\
\Delta P_1 &= P_1 \otimes \cos\left(\frac{\varrho}{2}P_0\right) + \cos\left(\frac{\varrho}{2}P_0\right) \otimes P_1 + P_2 \otimes \sin\left(\frac{\varrho}{2}P_0\right) - \sin\left(\frac{\varrho}{2}P_0\right) \otimes P_2, \\
\Delta P_2 &= P_2 \otimes \cos\left(\frac{\varrho}{2}P_0\right) + \cos\left(\frac{\varrho}{2}P_0\right) \otimes P_2 - P_1 \otimes \sin\left(\frac{\varrho}{2}P_0\right) + \sin\left(\frac{\varrho}{2}P_0\right) \otimes P_1, \\
\Delta M_{01} &= M_{01} \otimes \cos\left(\frac{\varrho}{2}P_0\right) + \cos\left(\frac{\varrho}{2}P_0\right) \otimes M_{01} + M_{02} \otimes \sin\left(\frac{\varrho}{2}P_0\right) - \sin\left(\frac{\varrho}{2}P_0\right) \otimes M_{02} \\
&\quad - P_1 \otimes \frac{\varrho}{2}M_{12}\cos\left(\frac{\varrho}{2}P_0\right) + \frac{\varrho}{2}M_{12}\cos\left(\frac{\varrho}{2}P_0\right) \otimes P_1 \\
&\quad - P_2 \otimes \frac{\varrho}{2}M_{12}\sin\left(\frac{\varrho}{2}P_0\right) - \frac{\varrho}{2}M_{12}\sin\left(\frac{\varrho}{2}P_0\right) \otimes P_2, \\
\Delta M_{02} &= M_{02} \otimes \cos\left(\frac{\varrho}{2}P_0\right) + \cos\left(\frac{\varrho}{2}P_0\right) \otimes M_{02} - M_{01} \otimes \sin\left(\frac{\varrho}{2}P_0\right) + \sin\left(\frac{\varrho}{2}P_0\right) \otimes M_{01} \\
&\quad - P_2 \otimes \frac{\varrho}{2}M_{12}\cos\left(\frac{\varrho}{2}P_0\right) + \frac{\varrho}{2}M_{12}\cos\left(\frac{\varrho}{2}P_0\right) \otimes P_2 \\
&\quad + P_1 \otimes \frac{\varrho}{2}M_{12}\sin\left(\frac{\varrho}{2}P_0\right) + \frac{\varrho}{2}M_{12}\sin\left(\frac{\varrho}{2}P_0\right) \otimes P_1, \\
\Delta M_{03} &= M_{03} \otimes 1 + 1 \otimes M_{03} - \frac{\varrho}{2}P_3 \otimes M_{12} + \frac{\varrho}{2}M_{12} \otimes P_3, \\
\Delta M_{12} &= M_{12} \otimes 1 + 1 \otimes M_{12}, \\
\Delta M_{13} &= M_{13} \otimes \cos\left(\frac{\varrho}{2}P_0\right) + \cos\left(\frac{\varrho}{2}P_0\right) \otimes M_{13} + M_{23} \otimes \sin\left(\frac{\varrho}{2}P_0\right) - \sin\left(\frac{\varrho}{2}P_0\right) \otimes M_{23} \\
\Delta M_{23} &= M_{23} \otimes \cos\left(\frac{\varrho}{2}P_0\right) + \cos\left(\frac{\varrho}{2}P_0\right) \otimes M_{23} - M_{13} \otimes \sin\left(\frac{\varrho}{2}P_0\right) + \sin\left(\frac{\varrho}{2}P_0\right) \otimes M_{13}.
\end{aligned}
$$

With this twist we can build a covariant $\star$ product, and field and gauge theories, as well as the Hopf algebra

$$\boxed{(f \star g)(x) = \mathcal{F}^{-1}(y,z) f(y) g(z)\Big|_{x=y=z} = fg - \frac{\mathrm{i}\varrho}{2}(\partial_\varphi f \partial_0 g - \partial_0 f \partial_\varphi g) + O(\varrho^2).}$$

which deforms the addition of momenta

$$\boxed{e^{-\mathrm{i}p\cdot x} \star e^{-\mathrm{i}q\cdot x} = e^{-\mathrm{i}(p +_\star q)\cdot x},}$$

$$\boxed{p +_\star q = R(q_0)p + R(-p_0)q,}$$

$$\boxed{R(t) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\left(\frac{\varrho t}{2}\right) & \sin\left(\frac{\varrho t}{2}\right) & 0 \\ 0 & -\sin\left(\frac{\varrho t}{2}\right) & \cos\left(\frac{\varrho t}{2}\right) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

With this is it possible to build a field theory. In particular we looked at $\boxed{\phi^4}$ Euclidean scalar theory. The usual arena to look for phenomena like ultraviolet/infrared mixing.

The deformed conservation of momenta gives a deforms the vertex, but not of the propagator.

On consequence is that decays are not anymore back to back. I have no time to go into this.

Final Remarks (and a postscript)

The main message I want to convey is that quantum gravity will require Quantum Spacetime.

Quantum Spacetime in turn requires quantum observers.

This is of course true for quantum phase space as well. There we became (more or less) used to deal with the contradictions of the quantum/classical interaction. We learned how to deal with noncommuting observables for example

But a quantum spacetime will pose further challenges and other layers to our understanding.

As postscript the rest of the acknowledgments of my thesis, a way to remember old friends:

Then Carl Rosenzweig. When I came to Syracuse my interests were mostly mathematical, I leave with a strong interest in phenomenology as well. This is mostly due to Carl's influence and I thank him for broadening my interests. The thought of doing physics without Vincent Rodgers is almost scary. I discussed and learned physics with him for five years in three continents. With Gianni Sparano I shared many a discussion of physics and a few Grand Crus. V.P. Nair and S.G. Rajeev never spared an explanation for the many things they know better than I do. Allen Stern never stopped checking my calculations (with very good reasons!)

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