From κ to ρ

Localizability, Quantization, Fields

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Happy Anniversary DSR!

In this audience I need not introduce $\boxed{\kappa}$-Minkowski, it is well known. I will discuss it together with a lesser know close relative:

$\boxed{\rho}$-Minkowski

$$\boxed{[x^0, x^1] = -\mathrm{i}\vartheta x^2 \ ; \ [x^0, x^2] = \mathrm{i}\vartheta x^1 \ ; \ [x^0, x^3] = 0 \ ; \ [x^i, x^j] = 0}$$

This form of noncommutativity was introduced by Amelino-Camelia, Barcaroli, Loret, Bianco and Pensato. They called $\boxed{\rho}$ what I call here $\boxed{\vartheta}$ for reasons which will be clear in a moment.

On the other side I also use $\boxed{\lambda}$ for $\boxed{\frac{1}{\kappa}}$...

A similar version can be built in which $\boxed{x^0}$ and $\boxed{x^3}$ are exchanged. I will discuss this variant in the field theory part.

Let me express the commutation relations in polar coordinates $\boxed{(t, r, \theta, \varphi)}$ for $\boxed{\kappa}$:

$$\boxed{[t, r] = \mathrm{i}\lambda r \ ; \ \text{``}[t, \varphi] = [t, \theta] = [r, \varphi] = [r, \theta] = 0\text{''}}$$

and for $\boxed{\rho}$ in cylindrical coordinates $\boxed{(t, \rho, z, \varphi)}$

$$\boxed{\text{``}[t, \varphi] = \mathrm{i}\vartheta\text{''} \, ; \ [t, z] = [t, \rho] = \text{``}[\rho, \varphi]\text{''} = [\rho, z] = 0}$$

Note that I have put some of the commutators in inverted commas.

I wish to study first the kinematics of this space. The non-commutative coordinates generate an algebra, and I will use the knowledge we developed for quantum phase space applied to this case.

In particular I will consider self-adjoint operators as observables, and represent the algebra generated by the coordinates on a Hilbert space, whose vectors will provide pure states. The possible results of a measurement are the points of the spectrum of the operators, the average is given by the expectation value etc.

This explains the inverted commas. The angular variables are not good observables, and a better expression would be one like $\boxed{[r, Y(\theta, \varphi)] = 0}$, where $\boxed{Y}$ is an operator generated by well defined functions of $\boxed{\theta, \varphi}$. In the interest of brevity I will sloppily sometimes use the angular variables as operators in formulas. In the papers everything is done in the proper way.

Except when comparing with the know case of quantum phase space, and for field theory at the end, $\boxed{\hbar}$ plays no role for $\boxed{\kappa}$ and $\boxed{\rho}$-Minkowski, this allows to consider $\boxed{t}$ as an operator without problems.

In quantum phase space, generated by $\boxed{[p_i, q^j] = \mathrm{i}\hbar\delta_i^j}$ there are two standard choices for a complete set of commuting observables, either positions $\boxed{\{q^i\}}$ or momenta $\boxed{\{p_i\}}$. Simultaneous measurement of both is prevented by Heisenberg uncertainty $\boxed{\Delta q \Delta p \geq \frac{\hbar}{2}}$.

It is possibile to represent the algebra generated by the six operators as acting on functions of $\boxed{q}$, with $\boxed{\psi(q)}$ multiplication operator and $\boxed{p}$ a differential operator, or act on functions of $\boxed{\tilde{\psi}(p)}$, with he role of $\boxed{p}$ and $\boxed{q}$ exchanged.

The two spaces are connected by a Fourier Transform.

Other basis are possible, for example $\boxed{N}$, number operator, square of angular momentum and one of its component. Any three self-adjoint mutually commuting will do, and identify a pure state with three real numbers.

For $\boxed{\kappa}$-Minkowski two complete sets are $\boxed{\{r, \theta, \varphi\}}$ or $\boxed{\{t, \theta, \varphi\}}$ and the uncertainty relation in polar coordinates reads

$$\boxed{\Delta t \Delta r \geq \frac{\lambda}{2} |\langle r \rangle|}$$

Since the angular variables commute, in the following I will ignore them.

It is possible to represent $\boxed{t}$ as operator on functions of $\boxed{r}$ as a dilation:

$$\boxed{t = \mathrm{i}\lambda \left(r\partial_r + \frac{3}{2} \right)}$$

The $\boxed{\frac{3}{2}}$ is necessary for self-adjointness.

Time has a continuous spectrum, and the (improper) eigenfunctions are the distributions

$$\boxed{T_\tau = \frac{r^{-\frac{3}{2}-\mathrm{i}\tau}}{\lambda^{-\mathrm{i}\tau}} = r^{-\frac{3}{2}} \mathrm{e}^{-\mathrm{i}\tau\log\left(\frac{r}{\lambda}\right)}}$$

Which play the role of plane waves for $\boxed{p}$.

The role played by Fourier transform previously is now played by a Mellin transform. A state can be written either as a function of $\boxed{r/\lambda}$, or of $\boxed{\tau = x^0/\lambda}$:

$$\boxed{\psi(r,\theta,\varphi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\tau\, r^{-\frac{3}{2}} \mathrm{e}^{-\mathrm{i}\tau\log\left(\frac{r}{\lambda}\right)} \widetilde{\psi}(\tau,\theta,\varphi),}$$

$$\boxed{\widetilde{\psi}(\tau,\theta,\varphi) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} r^2 \mathrm{d}r\, r^{-\frac{3}{2}} \mathrm{e}^{\mathrm{i}\tau\log\left(\frac{r}{\lambda}\right)} \psi(r,\theta,\varphi).}$$

The transformation is an isometry of $\boxed{L^2}$, $\boxed{|\psi|^2}$ and $\boxed{|\widetilde{\psi}|^2}$ are the probability densities to find the particle in position $\boxed{r}$ or time $\boxed{\tau}$ respectively

It is impossible to localise exactly a state both temporally and radially, except when the space is localised at $\boxed{r = 0}$.

The origin is the point at which the observer is located, and is not a “special point”. Another observer will be located at its own (different) origin, and will be able located states near to him.

$\boxed{\kappa}$-Minkowski is not Poincaré invariant, it is $\boxed{\kappa}$-Poincar invariant, and translations in this case are not commuting, therefore there is no contradiction in the fact that it is impossible for Alice to locate a state which Bob may. Alice cannot even precisely locate Bob!

All this is qualitatively perfectly compatible with the principle of relative locality, which however starts in a quite different context: curved momentum space. In this analysis instead momentum does not appear explicitly, although it is present in the symmetry.

Let us perform a similar analysis for $\boxed{\rho}$-Minkowski. This time the uncertainty will be between time and the angular variable. And one should definitely resist the temptation to write:

$$\boxed{\cancel{\Delta t \Delta \varphi \geq \frac{\vartheta}{2}}}$$

In the $\boxed{\{\rho, z, \varphi\}}$ basis $\boxed{t}$ is represented by the derivation operator $\boxed{-\mathrm{i}\vartheta\partial_\varphi}$.

This operator has Discrete Spectrum!

A change of basis is given by the Fourier series. And the eigenstates of momentum are $\boxed{\mathrm{e}^{\mathrm{i}n\varphi}}$, and they are completely delocalised in $\boxed{\varphi}$

On the other hand, a state completely localised in $\boxed{\varphi}$, given by a $\boxed{\delta}$, which requires a superposition with equal weights of all eivenvalues of time.

$$\boxed{\delta(\varphi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\varphi}}$$

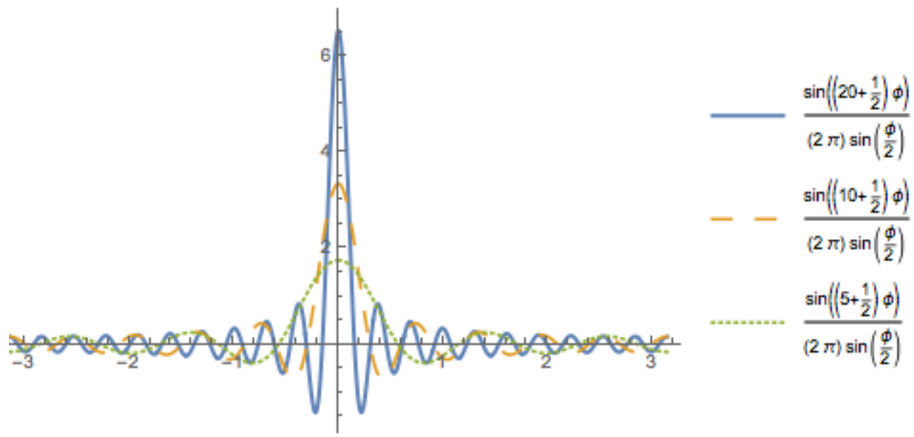
After a time measurement, which has given as result $\boxed{n_0\vartheta}$, the system is in the eigenstate $\boxed{\mathrm{e}^{\mathrm{i}n_0\varphi}}$.

A slightly uncertain state uses a great number of Fourier modes to built a state peaked around some time, then the corresponding uncertainty is the angular variable is given by the fact that only a finite set of elements of the basis are available.

For $\boxed{\vartheta}$ Planckian of the quantum of time (also called a chronon), is $\boxed{5.39\ 10^{-44}\, sec.}$

The most accurate measurement of time is $\boxed{\sim 10^{-19}\,sec.}$ Heuristically the superposition of $\boxed{10^{35}}$ quanta of time is needed.

Approximate $\boxed{\delta}$ by the Dirichlet nucleus $\boxed{\delta_N = \sum_{n=-N}^{N} e^{in\varphi} = \frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})\varphi}{\sin\frac{N}{2}\varphi}}$



For $\boxed{N = 5, 10, 15}$.

This most precise experiment needs $\boxed{N \sim 10^{35}}$. Then the first zero of the nucleus is at $\boxed{\varphi \sim 10^{-35}}$. We may assume this to be the uncertainty in an angle determination. To translate this as an uncertainty in position we need $\boxed{\rho}$. For the radius observable universe ($\boxed{10^{26}\,m}$) the uncertainty is $\boxed{10^{-9}\,m}$.

Is this all pervading clicking a feature of our universe? Is time translation definitely lost? Putting time on a lattice may be disturbing.

Self-adjointness come to the rescue. Anybody who has studied the Aharonov-Bohm experiment knows that the momentum operator on a compact domain is a rich operator.

It is self-adjoint on periodic functions, but is also selfadjoint on functions periodic up to a phase. In this case the eigenfunctions are $\boxed{\mathrm{e}^{\mathrm{i}(n+\alpha)\varphi}}$.

The differences between states is unchanged, and the effect is a rigid shift. This however means that a different choices of selfadjointess domains. Time translations are undeformed, and two time translated observers will be in different, but equivalent domains.

Just as in the the previous case, $\boxed{\rho}$-Minkowski is not Poincaré invariant. It is invariant under a $\boxed{\rho}$-Poincaré Hopf algebra.

Noticing that $\boxed{[\partial_t, \partial\varphi] = 0}$, the deformation can be built with a Drinfeld twist.

$$\boxed{\mathcal{F}(x,y) = \exp\left\{-\frac{\mathrm{i}\vartheta}{2}\left(\partial_{y^0}\left(x^2\partial_{x^1} - x^1\partial_{x^2}\right) - \partial_{x^0}\left(y^2\partial_{y^1} - y^1\partial_{y^2}\right)\right)\right\}}$$

$$\boxed{= \exp\left\{\frac{\mathrm{i}\vartheta}{2}\left(\partial_{y^0}\partial_{\varphi_x} - \partial_{x^0}\partial_{\varphi_y}\right)\right\}}$$

This deforms the Hopf algebra as

$$
\begin{aligned}
\Delta P_3 &= P_3 \otimes 1 + 1 \otimes P_3, \\
\Delta P_0 &= P_0 \otimes 1 + 1 \otimes P_0, \\
\Delta P_1 &= P_1 \otimes \cos\left(\frac{\vartheta}{2}P_0\right) + \cos\left(\frac{\vartheta}{2}P_0\right) \otimes P_1 + P_2 \otimes \sin\left(\frac{\vartheta}{2}P_0\right) - \sin\left(\frac{\vartheta}{2}P_0\right) \otimes P_2, \\
\Delta P_2 &= P_2 \otimes \cos\left(\frac{\vartheta}{2}P_0\right) + \cos\left(\frac{\vartheta}{2}P_0\right) \otimes P_2 - P_1 \otimes \sin\left(\frac{\vartheta}{2}P_0\right) + \sin\left(\frac{\vartheta}{2}P_0\right) \otimes P_1, \\
\Delta M_{01} &= M_{01} \otimes \cos\left(\frac{\vartheta}{2}P_0\right) + \cos\left(\frac{\vartheta}{2}P_0\right) \otimes M_{01} + M_{02} \otimes \sin\left(\frac{\vartheta}{2}P_0\right) - \sin\left(\frac{\vartheta}{2}P_0\right) \otimes M_{02} \\
&\quad - P_1 \otimes \frac{\vartheta}{2}M_{12} \cos\left(\frac{\vartheta}{2}P_0\right) + \frac{\vartheta}{2}M_{12} \cos\left(\frac{\vartheta}{2}P_0\right) \otimes P_1 \\
&\quad - P_2 \otimes \frac{\vartheta}{2}M_{12} \sin\left(\frac{\vartheta}{2}P_0\right) - \frac{\vartheta}{2}M_{12} \sin\left(\frac{\vartheta}{2}P_0\right) \otimes P_2, \\
\Delta M_{02} &= M_{02} \otimes \cos\left(\frac{\vartheta}{2}P_0\right) + \cos\left(\frac{\vartheta}{2}P_0\right) \otimes M_{02} - M_{01} \otimes \sin\left(\frac{\vartheta}{2}P_0\right) + \sin\left(\frac{\vartheta}{2}P_0\right) \otimes M_{01} \\
&\quad - P_2 \otimes \frac{\vartheta}{2}M_{12} \cos\left(\frac{\vartheta}{2}P_0\right) + \frac{\vartheta}{2}M_{12} \cos\left(\frac{\vartheta}{2}P_0\right) \otimes P_2 \\
&\quad + P_1 \otimes \frac{\vartheta}{2}M_{12} \sin\left(\frac{\vartheta}{2}P_0\right) + \frac{\vartheta}{2}M_{12} \sin\left(\frac{\vartheta}{2}P_0\right) \otimes P_1, \\
\Delta M_{03} &= M_{03} \otimes 1 + 1 \otimes M_{03} - \frac{\vartheta}{2}P_3 \otimes M_{12} + \frac{\vartheta}{2}M_{12} \otimes P_3, \\
\Delta M_{12} &= M_{12} \otimes 1 + 1 \otimes M_{12}, \\
\Delta M_{13} &= M_{13} \otimes \cos\left(\frac{\vartheta}{2}P_0\right) + \cos\left(\frac{\vartheta}{2}P_0\right) \otimes M_{13} + M_{23} \otimes \sin\left(\frac{\vartheta}{2}P_0\right) - \sin\left(\frac{\vartheta}{2}P_0\right) \otimes M_{23} \\
\Delta M_{23} &= M_{23} \otimes \cos\left(\frac{\vartheta}{2}P_0\right) + \cos\left(\frac{\vartheta}{2}P_0\right) \otimes M_{23} - M_{13} \otimes \sin\left(\frac{\vartheta}{2}P_0\right) + \sin\left(\frac{\vartheta}{2}P_0\right) \otimes M_{13}.
\end{aligned}
$$

With this twist we can build a covariant $\star$ product, and field and gauge theories, as well as the Hopf algebra

$$\boxed{(f\star g)(x) = \mathcal{F}^{-1}(y,z)f(y)g(z)\Big|_{x=y=z} = fg - \frac{\mathrm{i}\vartheta}{2}(\partial_\varphi f\partial_0 g - \partial_0 f\partial_\varphi g) + O(\vartheta^2).}$$

which deforms the addition of momenta

$$\boxed{e^{-\mathrm{i}p\cdot x}\star e^{-\mathrm{i}q\cdot x} = e^{-\mathrm{i}(p+_\star q)\cdot x},}$$

$$\boxed{p +_\star q = R(q_0)p + R(-p_0)q,}$$

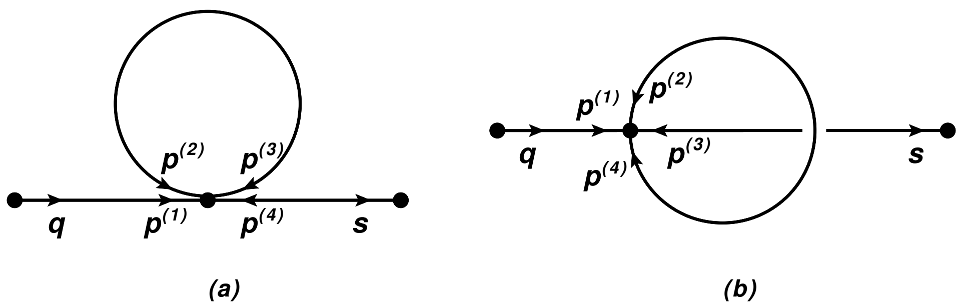
$$\boxed{R(t) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\left(\frac{\vartheta t}{2}\right) & \sin\left(\frac{\vartheta t}{2}\right) & 0 \\ 0 & -\sin\left(\frac{\vartheta t}{2}\right) & \cos\left(\frac{\vartheta t}{2}\right) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

With this is it possible to build a field theory. In particular we looked at $\boxed{\phi^4}$ Euclidean scalar theory. The usual arena to look for phenomena like ultraviolet/infrared mixing.

The deformed conservation of momenta gives a deformation of the vertex but not of the propagator. This is because the $\boxed{\delta}$ of conservation of momentum behaves like;

$$\boxed{\delta^{(4)}\left(p^{(1)} +_\star \ldots +_\star p^{(k)} +_\star \ldots +_\star p^{(N)}\right) = \delta^{(4)}\left(p^{(1)} +_\star \ldots + p^{(k)} +_\star \ldots +_\star p^{(N)}\right)}$$

We calculated the one loop corrections to the propagator, which are different for the planar and nonplanar cases, and often the latter exhibits mixing.



At the Euclidean level it does not matter if the noncommuting variable is $\boxed{x^3}$ ot $\boxed{x^0}$, but the Wick rotation will be different in the two cases, because of the nature of the matrix $\boxed{R}$

I will just enumerate the results of the calculations.

- The planar diagrams are unchanged

- The non planar diagrams are modified. There is a difference between the $\boxed{x^3}$ or $\boxed{x^0}$ case. The former exhibits a softening of the ultraviolet divergences, just as in the Moyal case.

- For $\boxed{x^0}$ noncommutative the integration over $\boxed{p_0}$ develops a singularity.

- Nonplanar contributions explode for $\boxed{\vartheta \to 0}$. The amplitude explodes also for any choice of $\boxed{\vartheta}$ and $\boxed{q_3}$ such that $\boxed{\vartheta q_3 = k\ 2\pi}$ with $\boxed{k}$ arbitrary integer. The situation is quite typical for the UV/IR mixing: the $\boxed{\vartheta \to 0}$ limit does not commute with the large cutoff asymptotics, in particular the latter does not exhibit a smooth commutative limit.

- The nonplanar correction to the propagator is not proportional to $\boxed{\delta(q - s}$, what implies that the "deformed momentum conservation law" which holds at the classical level is anomalously broken by quantum corrections.

- In $\boxed{2+1}$ dimensions things are much better, but the UV/IR mixing persists

- It is possible to examine the particle decay. For the $\boxed{x^3}$ variant for particles at rest there is no change, while for particles in motion the decay is deformed, and not "back to back". For a particle in the $\boxed{3}$ direction, the angle in $\boxed{12}$ plane is $\boxed{\Delta\varphi = \pi - \frac{\vartheta p_3}{2}}$

- For the $\boxed{x^0}$ variant there is a deformation also for particles at rest. In this case the angle among the particles is $\boxed{\Delta\varphi = \pi - \frac{\vartheta M}{2}}$

Final Remarks

$\boxed{\rho}$-Minkowski has some nice features, but it is not clear if its singling out of a direction in space, as it happens for Moyal, is compatible with physics. Surveys give stringent limits.

Nevertheless, it is still necessary to study the transformation among observers, and see if really there is such an invariant universal direction, or if a proper use of the quantum group of transformations solves the problem.

What I personally find most intriguing is the possibility that some form of quantum time may give a discrete structure emerging. Obviously cannot be a lattice, but possibly operators with discrete spectra may be interesting. This is not new, there have been examples in loop quantum gravity, and this little contribution goes in that direction.