Points, lack thereof

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In my elementary school book a **point** was defined as:

A geometrical Entity without Dimension

I must confess that after reading it I was none the wiser about what a point is

Probably because I was convinced I knew what a point is. I could produce them at will with my biro. Or better with a sharper pin, or better. . .

Euclid defined it as **That which has no part**

Points are ubiquitous in both physics and mathematics, but are we sure we are always talking of the same thing?

In astrophysics a point may be a galaxy, or even a cluster of galaxies

In general relativity a point is an event. Which implies some structure anyway

In classical "point particle" dynamics we use points of phase space to describe the state of motion of a particle

Quantum mechanics: phase space changes dramatically, becomes a **Noncommutative Geometry**.

Position and momentum, become noncommuting operators on a Hilbert space. The closest you may get to the concept of points are coherent states.

I will not dwell further on quantum phase space, in the rest of this talk I will be concerned with ordinary (configuration) space, and spacetime.

In quantum mechanics space has no problem. Position operators commute among themselves, and the algebra they generate has pure states which can be associated to points.

If we attempt to define points in space(time) at very short distance we run into trouble if we put together quantum mechanics and gravity.

There is a phenomenon noticed probably for the first time by Bronstein in 1938, but presented independently in a modern and most terse way by Doplicher, Fredenhagen and Roberts in 1994.

I will present a caricature of these arguments, which however captures the main idea in a nontechnical way.

It is a variant of the Heisenberg microscope justification of the uncertainty principle

The idea is that to "see" something small, of size of the order of $\boxed{\Delta x}$, we have to send a "small" photon, that is a photon with a small wavelength $\boxed{\lambda}$, but a small wavelength means a large momentum $\boxed{p = h/\lambda}$. In the collision there will a transfer of momentum, so that we can capture the photon. The amount of momentum transferred is uncertain.

If one does the calculation using the resolving power of an ideal microscope one finds:

$$\boxed{\Delta x \Delta p \geq h}$$

where $\boxed{h}$ is Planck's constant.

The argument is very heuristic, and the result is off by an order of magnitude ($\boxed{4\pi}$). We know that in order to obtain the uncertainty principle it is necessary to have a solid theory, quantum mechanics, where $\boxed{p}$ and $\boxed{q}$ become operators, and then it is possible to prove:

$$\boxed{\Delta x \Delta x \geq \frac{\hbar}{2}}$$

We are interested only is space, and not momentum, for which there is no limitation in quantum mechanics to an arbitrary precise measurement of $\boxed{x}$

If we include gravity in the game things change. We now have a length scale obtained combining the speed of light, Planck's constant and Newton's constant: $\boxed{\ell = \sqrt{\frac{\hbar G}{c^3}} \simeq 10^{-33}\text{cm}}$

In order to "measure" the position of an object, and hence the "point" in space, one has use a very small probe, which has to be very energetic, but on the other side general relativity tells us that if too much energy is concentrated in a small region a black hole is formed.

DSR obtained the following relation:

$$\boxed{\Delta x_0(\Delta x_1 + \Delta x_2 + \Delta x_3) \geq \ell^2} \qquad \boxed{\Delta x_1 \Delta x_2 + \Delta x_2 \Delta x_3 + \Delta x_1 \Delta x_3 \geq \ell^2}$$

For a rigorous statement we would need a full theory of **quantum gravity**. A theory which do not (hopefully yet) posses.

Note that more than the points, we need the correlation among them. The mathematical result of Gelfand and Naimark is not only that a commutative C^*-algebra provides a set of points, but that one may also infer topology, i.e. when a sequence of points converges to another point.

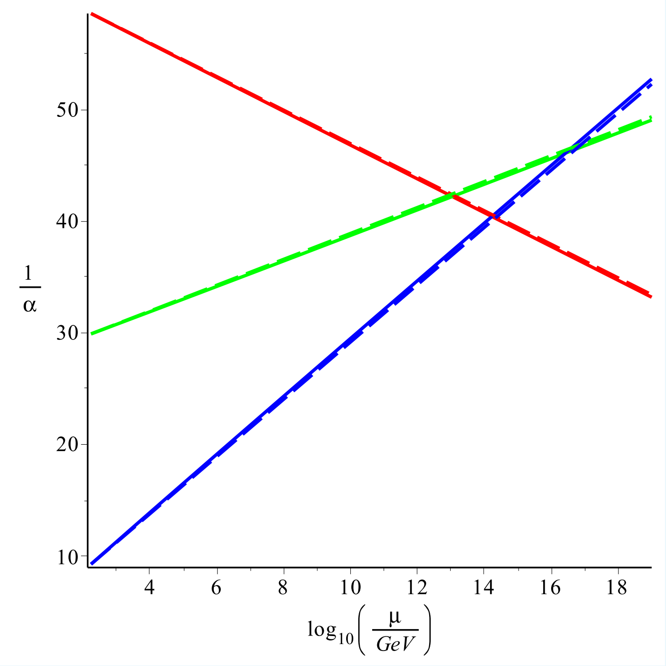
I will now try to use the most advanced theory which encompasses relativity and quantum mechanics, quantum field theory, to infer what the relation among points are at very high energy

I will use the knowledge form field theory at energies below the (yet to be defined) transition scale at which quantum geometry appears , to infer some knowledge of quantum spacetime

I will be in a definite context, that of spectral geometry, and expecially the spectral action, but the reasoning I will make is quite general.

The way one can learn what happens beyond the scale of an experiment is to use the renormalization flow of the theory

We know that the coupling constants, i.e. the strength of the interaction, change with energy.



This picture is valid in the absence of new physics, i.e. new particles and new interactions which would alter the equations which govern the running

The three interaction strength start from rather different values but come together almost at a single unification point

But then the nonabelian interactions proceed towards asymptotic freedom, while the abelian one climbs towards a Landau pole at incredibly high energies $\boxed{10^{53}\ \text{GeV}}$

The lack of a unification point was one of the reasons for the falling out of fashion of GUT's.

Some supersymmetric theories have unification point

We all believe that this running will be stopped by "something" at $\boxed{10^{19}\ \text{GeV}}$

This unknown something we call quantum gravity

I take the point of view that there is a fundamental change of the degrees of freedom of spacetime. One useful tool to describe this is Noncomutative Spectral Geometry

The metric and geometric properties are encoded in the (generalized) Dirac operator $\boxed{D}$ which fixes the background around which expand the action

The eigenvalues of the Dirac operator on a curved spacetime are diffeomorphism-invariant functions of the geometry. They form an infinite set of observables for general relativity.

The interaction among fields is described by the Spectral Action

$$S = \mathrm{Tr}\,\chi\left(\frac{D_A^2}{\Lambda^2}\right)$$

χ is a cutoff function, which we may take to be a decreasing exponential or the characteristic function of the interval

$D_A = D + A$ is a fluctuation of the Dirac operator, A a connection one-form built from D as $A = \sum_i a_i[D, b_i]$ with a, b elements of the algebra, the fluctuations are ultimately the variables, the fields of the action

Λ is a cutoff scale without which the trace would diverge.

The spectral action can be expanded in powers of Λ^{-1} using standard heat kernel techniques

In this framework it is possible to describe the action of the standard model

One has to choose as $\boxed{D}$ operator the tensor product of the usual Dirac operator on a curved background $\boxed{\nabla\!\!\!/}$ times a matrix containing the fermionic parameters of the standard model (Yukawa couplings and mixings), acting on the Hilbert space of fermions

In this way one "saves" one parameter, and can predict the mass of the Higgs. The original prediction was $\boxed{170 \text{ GeV}}$, which is not a bad result considering that the theory is basically based on pure mathematical requirements

When it was found at $\boxed{125 \text{ GeV}}$ it was realized that the model had to be refined (right handed neutrinos play a central role) to make it compatible with present experiments. (Stephan, Devastato Martinetti, FL, Chamseddine, Connes, Van Suijlekom

But this is another seminar. . .

Let us analyse the role of Λ. Without it, the trace diverges. Field theory cannot be valid at all scales. It is itself a theory which emerges form a yet unknown quantum gravity

This points to a geometry in which the spectrum of operators like Dirac operator are truncated, i.e. the eigenvalues "saturate" at Λ, which appears as the top scale at which one can use QFT. One may identify this scale with ℓ, but it might be different (even lower, at the unification scale).

Consider the eigenvectors $|n\rangle$ of D in increasing order of the respective eigenvalue λ_n. $D = \sum_0^\infty |n\rangle \lambda_n \langle n|$.

Define N as the maximum value for which $\lambda_n \leq \Lambda$. This defines the truncated Dirac operator $\sum_0^N |n\rangle \lambda_n \langle n| + \sum_N^\infty |n\rangle \Lambda \langle n|$.

We are effectively saturating the operator at a scale Λ

Given a space with a Dirac operator one can define a distance (Connes) between states of the algebra of functions, in particular points are (pure) states and the distance is:

$$\boxed{d(x,y) = \sup_{\|[D,f]\|\leq 1} |f(x) - f(y)|}$$

It is possible possible to prove D'Andrea, FL, Martinetti that using $\boxed{D_\Lambda}$ the distance among points is infinite

In general for a bounded Dirac operator of norm $\boxed{\Lambda}$ then $\boxed{d(,x,y) > \Lambda^{-1}}$, and to find states at finite distance one has to consider "extended" points, such as coherent spates

The role of a truncated Dirac operator, or in general wave operator, has been introduced in field theory even before the spectral action. It goes under the name of Finite Mode Regularization,

Andrianov, Bonora, Fujikawa

Consider the generic fermionic action:

$$Z = \int [d\bar{\psi}][d\psi] e^{-\langle\psi|D|\psi\rangle} \overset{\text{formally}}{=} \det D$$

The equality is formal because the expression is divergent, and has to be regularized, for example considering D_Λ

One can study the renormalization flow, and note that the measure is not invariant under scale transformation, giving rise to a potential anomaly Andrianov, Kurkov, FL

The induced term by the flow, which takes care of the anomaly, turn out to be exactly the spectral action

The hypothesis is that $\boxed{\Lambda}$ has a physical meaning, it is a scale indicating a phase transition, and we can try to infer some properties of the phase above $\boxed{\Lambda}$ studying the high energy limit of the action with the cutoff.

At high momentum Green's function, the inverse of $\boxed{D_\Lambda}$, effectively is the identity in momentum space

I will now see this in greater detail considering the bosonic sector

Usually probes are bosons, hence let me consider the expansion of the spectral action in the high momentum limit Kurkov, FL, Vassilevich

This has been made by Barvinsky and Vilkovisky who were able to sum all derivatives (for a decreasing exponential cutoff function):

$$\text{Tr}\ \exp\left(-\frac{D^2}{\Lambda^2}\right) \simeq \frac{\Lambda^4}{(4\pi)^2} \int d^4x \sqrt{g}\, \text{tr}\ \Big[1 + \Lambda^{-2} P +$$

$$\Lambda^{-4} \left(R_{\mu\nu} f_1 \left(-\frac{\nabla^2}{\Lambda^2}\right) R^{\mu\nu} + R f_2 \left(-\frac{\nabla^2}{\Lambda^2}\right) R +\right.$$

$$\left. P f_3 \left(-\frac{\nabla^2}{\Lambda^2}\right) R + P f_4 \left(-\frac{\nabla^2}{\Lambda^2}\right) P + \Omega_{\mu\nu} f_5 \left(-\frac{\nabla^2}{\Lambda^2}\right) \Omega^{\mu\nu}\right)\Big] + O(R^3, \Omega^3, E^3)$$

where $P = E + \frac{1}{6} R$ and $f_1, \ldots, f_5$ are known functions, high momenta asymptotic of form factor:

$f_1 ... f_5$ read:

$$\begin{aligned}
f_1(\xi) &\simeq \frac{1}{6}\xi^{-1} - \xi^{-2} + O\left(\xi^{-3}\right) \\
f_2(\xi) &\simeq -\frac{1}{18}\xi^{-1} + \frac{2}{9}\xi^{-2} + O\left(\xi^{-3}\right) \\
f_3(\xi) &\simeq -\frac{1}{3}\xi^{-1} + \frac{4}{3}\xi^{-2} + O\left(\xi^{-3}\right) \\
f_4(\xi) &\simeq \xi^{-1} + 2\xi^{-2} + O\left(\xi^{-3}\right) \\
f_5(\xi) &\simeq \frac{1}{2}\xi^{-1} - \xi^{-2} + O\left(\xi^{-3}\right)
\end{aligned}$$

Let me consider a Dirac operator containing just the relevant aspects, i.e. a bosonic fields and the fluctuations of the metric.

$$\not{D} = i\gamma^{\mu}\nabla_{\mu} + \gamma_5\phi = i\gamma^{\mu}(\partial_{\mu} + \omega_{\mu} + iA_{\mu}) + \gamma_5\phi$$

with ω_{μ} the Levi-Civita connection and $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$

It is now possible to perform the B-V expansion to get the expression for the high energy spectral action

$$S_B \simeq \frac{\Lambda^4}{(4\pi)^2}\int d^4x \left[-\tfrac{3}{2}h_{\mu\nu}h_{\mu\nu} + 8\phi\frac{1}{-\partial^2}\phi + 8F_{\mu\nu}\frac{1}{(-\partial^2)^2}F_{\mu\nu}\right]$$

In order to understand the meaning of this action let me remind how we get propagation of waves and correlation of points in usual QFT with action

$$S[J,\varphi] = \int d^4x \left[\varphi(x)\left(\partial^2 + m^2\right)\varphi(x) - J(x)\varphi(x)\right]$$

To this correspond the equation of motion

$$\left(\partial^2 + m^2\right)\varphi(y) = J(y)$$

And the Green's function $G(x-y)$ which "propagates" the source:

$$\varphi_J(x) = \int d^4y J(y) G(x-y)$$

In momentum representation we have

$$\varphi(x) = \frac{1}{(2\pi)^2} \int d^4k \; e^{ikx} \; \widehat{\varphi}(k)$$

$$J(x) = \frac{1}{(2\pi)^2} \int d^4k \; e^{ikx} \; \hat{J}(k)$$

$$G(x-y) = \frac{1}{(2\pi)^2} \int d^4k \; e^{ik(x-y)} \; \widehat{G}(k)$$

And the propagator is

$$G(k) = \frac{1}{\left(k^2 + m^2\right)}$$

The field at a point depends on the value of field in nearby points, and the points "talk" to each other exchanging virtual particles

In the general case of a generic boson φ, the Higgs, an intermediate vector boson or the graviton and $F(\partial^2)$ the appropriate wave operator, a generalised Laplacian

$$S[J,\phi] = \int d^4x \left(\frac{1}{2}\varphi(x)F(\partial^2)\varphi(x) - J(x)\varphi(x)\right) \ ,$$

In this case the equation of motion is $F(\partial^2)\phi(x) = J(x)$ giving

$$G = \frac{1}{F(\partial^2)} \quad , \quad G(k) = \frac{1}{F(-k^2)}$$

and

$$\varphi_J(x) = \int d^4y J(y) G(x-y) = \frac{1}{(2\pi)^4}\int d^4k e^{ikx} J(k)\frac{1}{F(-k^2)}$$

The cutoff is telling us that

$$\varphi_J(x) = \int d^4y J(y) G(x-y) = \frac{1}{(2\pi)^4} \int d^4k e^{ikx} J(k) \frac{1}{F(-k^2)}$$

The short distance behaviour is given by the limit $k \to \infty$

Consider $J(k) \neq 0$ for $|k^2| \in [K^2, K^2 + \delta k^2]$, with K^2 very large.

$$\varphi_J(x) \xrightarrow[K\to\infty]{} \begin{cases} \frac{1}{(2\pi)^4} \int d^k e^{ikx} J(k) k^2 = (-\partial^2) J(x) & \text{for scalars and vectors} \\ \frac{1}{(2\pi)^4} \int d^k e^{ikx} J(k) = J(x) & \text{for gravitons} \end{cases}$$

This corresponds to a limit of the Green's function in position space

$$G(x-y) \propto \begin{cases} (-\partial^2)\delta\,(x-y) & \text{for scalars and vectors} \\ \delta\,(x-y) & \text{for gravitons} \end{cases}$$

The correlation vanishes for noncoinciding points, heuristically, nearby points "do not talk to each other".

This is a limiting behaviour, I think one has to take it as a general indication that the presence of a physical cutoff scale in momenta leads to a "non geometric phase" in which the concept of point ceases to have meaning, possibly described by a noncommutative geometry

Note that throughout this discussion I have done nothing to spacetime, I have only imposed the cutoff and used standard techniques and interpretations

Points may still be there, but they are uncorrelated at very high energy.

Clearly this indicates a phase transition for which the locality order parameter (position) is not present anymore.

But it also gives us the indication that quantum gravity must be a theory in which points are not a relevant entity.

This is coherent with all other indications.

It is like a deep water fish trying to understand what goes on above his ceiling.

He knows that pressure decreases as he goes up. He can also infer some properties of a different states of matter by looking at bubbles which are creates near some "high energy" volcanic vents or when "above" there are storms, but he cannot naturally grasp the concept of air, or absence of water.

He will need a higher leap