Localization and Reference Frames

in κ-Minkowski Spacetime

Fedele Lizzi

Università di Napoli Federico II

and Institut de Ciencies del Cosmos, Universitat de Barcelona

work in collaboration with M. Manfredonia, F. Mercati and T. Poulain
arXiv:1811.08409

Kolkata 2018

I will discuss the quantum spacetime describe by the κ-Minkowski commutation relations:

$$[x^0, x^i] = \mathrm{i}\lambda x^i, \ \ [x^i, x^j] = 0, \ \ i, j = 1, 2, 3.$$

This is a noncommutative geometry, like the quantum phase space. I will ask what are the possible measurements, the states, their localisation, change of observers. . .

I will be very basic and follow Diracs correspondence principle, associating to coordinates and all observables operators on a Hilbert space. Consider their spectrum, eigenfunctions and spectral decomposition.

I assume that the eigenvalues are the possible results of a measurement of the observables, and use the standard apparatus of quantum mechanics, but not consider conjugate momenta and their commutations. $\hbar$ plays no role, except when comparing with quantum phase space.

Let me first introduce a case study: Quantum Phase Space of a particle.

This is of course well known material, but I will use it to compare with κ-Minowski.

Phase space is a six-dimensional space spanned by (q^i, p_i). Quantization introduces the commutation relation $[q^i, p_j] = \mathrm{i}\hbar\delta^i_j$,

The most common representations of position and momenta is operators on $L^2(\mathbb{R}^3_q)$

$$\widehat{q}^i\psi(q) = q^i\psi(q) \ ; \qquad \widehat{p}_i\psi(q) = -\mathrm{i}\hbar\frac{\partial}{\partial q^i}\psi(q) \ .$$

$\widehat{q}$'s and $\widehat{p}$'s are unbounded selfadjoint operators with a dense domain. The spectrum is the real line (for each i).

They have no eigenvectors but improper eigenfunctions: distribution.

Since the $\widehat{q}^i$'s commute it is possible to have a simultaneous improper eigenvector of all of them, these are the Dirac distributions $\delta(q-\bar{q})$ for a particular $\bar{q}$ vector in $\mathbb{R}^3$ For a particular momentum $\bar{p}$ the improper eigenfunctions of the $\widehat{p}_i$ are plane waves $e^{i\bar{p}_i q^i}$.

Formally, the eigenvalue equation $\partial_q \psi(q) = \alpha \psi(q) \ , \alpha \in \mathbb{C}^3$ is solved by $e^{\alpha \cdot q}$ with a vector α

No function of this kind is square integrable, there are no (proper) eigenfunctions. The operator $\widehat{p}$ is self-adjoint on the domain of absolutely continuous functions. α must be pure imaginary because the distributions must be well defined on the domain of selfadjointness of the operators.

The improper eigenfunctions of momentum are physically interpreted as infinite plane waves of precise frequency. Since plane waves are not vectors of the Hilbert space there is no quantum state which would give as measure exactly the value $\hbar k$, nevertheless we have all learned to live with this fact, and there is a well-defined sense in which we talk about "particles of momentum $\boxed{\hbar k}$".

Implicitly we have chose $\boxed{\widehat{q}^i}$ as a complete set of observables, the description of a quantum state as a function of positions. $\boxed{|\psi(q)|^2}$ (normalized) is the density probability to find the particle at position $\boxed{q}$.

The $\boxed{\psi}$ is complex and contains also the information about the density probability of the momentum operator.

We could have chosen $\boxed{\widehat{p}}$ as complete set. Then we would have the Fourier transformed $\boxed{\phi(p)}$ It is important that the Fourier transform is an isometry, it maps normalized functions of positions into normalized functions of momenta

And of course we have other choices, number operator and angular momentum . . .

This was for quantum phase space, and the deformation parameter was $\boxed{\hbar}$

I now want to reproduce this discussion for $\boxed{\kappa}$-Minkowski, four dimensionlal space with different commutation relation and a deformation parameter $\boxed{\lambda = \frac{1}{\kappa}}$

This is a quantum space, but I will only consider its kinematic, and leave $\boxed{\hbar}$ alone for the moment.

But is a relativistic space, hence later I will worry about Poincaré transformations.

I will look for a representation of the x^μ on $L^2(\mathbb{R}^3)$:

$$\hat{x}^i \psi(x) = x^i \psi(x)$$

$$\hat{x}^0 \psi(x) = \mathrm{i}\lambda \left(\sum_i x^i \partial_{x^i} + \frac{3}{2} \right) \psi(x) = \mathrm{i}\lambda \left(r\partial_r + \frac{3}{2} \right) \psi(x).$$

Positions are multiplicative operators, time is dilation. The $3/2$ factor is necessary to make the operator symmetric.

For dilations the polar basis is appropriate. The commutation relationsand uncertainty principle become

$$[\hat{x}^0, \cos\theta] = [\hat{x}^0, \mathrm{e}^{\mathrm{i}\varphi}] = 0 \ , [x^0, r] = \mathrm{i}\lambda r.$$

$$\Delta x^0 \Delta r \geq \frac{\lambda}{2} |\langle r \rangle|.$$

The operator is selfadjoint on all absolutely continuous functions

What is the spectrum of the time operator? Monomial in r are formal solutions of the eigenvalue problem:

$$\boxed{\mathrm{i}\lambda\left(r\partial_r + \frac{3}{2}\right) r^\alpha = \mathrm{i}\lambda(\alpha + \frac{3}{2})r^\alpha = \lambda_\alpha r^\alpha,}$$

The eigenvalues are real if and only if $\boxed{\alpha = -\frac{3}{2} + \tau}$ with $-\infty < \tau < \infty$ a real number.

For momentum we had plane waves, in this case we have the following distributions

$$\boxed{T_\tau = \frac{r^{-\frac{3}{2}-\mathrm{i}\tau}}{\lambda^{-\mathrm{i}\tau}} = r^{-\frac{3}{2}}\mathrm{e}^{-\mathrm{i}\tau\log(\frac{r}{\lambda})}}$$

The distribution has the correct dimension of a length to the 3/2, the factor of $\boxed{\lambda}$ is there to avoid taking the logarithm of a dimensional quantity. Since $\boxed{\lambda}$ is a natural scale for the model, its choice is natural, but not unique.

For quantum phase space we had as complete set of observables either three q ot three p, and we could back and forth among the two by a Fourier transofrm,

For κ-Minkowski we have either (r,θ,φ) or (τ,θ,φ), and we switch among the two with a Mellin transform

$$\psi(r,\theta,\varphi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}\tau\, r^{-\frac{3}{2}}\mathrm{e}^{-\mathrm{i}\tau\log\left(\frac{r}{\lambda}\right)}\widetilde{\psi}(\tau,\theta,\varphi)=\mathcal{M}^{-1}\left[\widetilde{\psi}(\tau,\theta,\varphi),\ r\right],$$

$$\widetilde{\psi}(\tau,\theta,\varphi)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\mathrm{d}r\, r^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\tau\log\left(\frac{r}{\lambda}\right)}\psi(r,\theta,\varphi)=\mathcal{M}\left[\psi(r,\theta,\varphi),\frac{3}{2}+\mathrm{i}\tau\right].$$

$|\psi|^2$ and $|\widetilde{\psi}|^2$ can be interpreted as the probabilty density to find the particle in position r or time τ respectively

It is useful to have an idea of the dimensional quantities involved.

Call t the eigenvalue of the time operator $\frac{x^0}{c}$, then $\tau = t\frac{c}{\lambda}$.

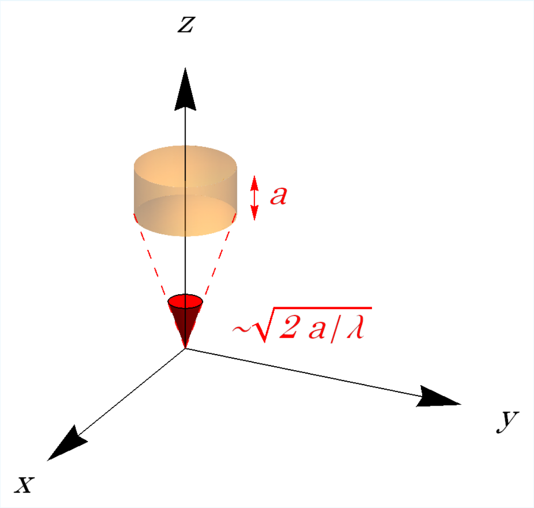
$\frac{c}{\lambda}$ is a dimensional quantity. If we choose for λ the Planck length then $\frac{c}{\lambda} \sim 2 \cdot 10^{43}$ Hz. In other words if $t = 1$ s, then $\tau = 2 \cdot 10^{43}$, an extremely large number.

If t is of the order of Planck time, then $\tau \sim 1$.

I will now give some examples of localised state, at the origin and away

Consider the following state (chosen to simplify calculations) localised in space in a small region of size a around a point at distance z_0 along the z axis.

$$\psi_{z_0,a}(r,\theta,\varphi) = \begin{cases} \sqrt{\frac{3\lambda}{2a\pi((a+z_0)^3 - z_0^3)}}, & z_0 \leq r \leq (z_0 + a) \text{ and } \cos\theta > 1 - \frac{a}{\lambda} \\ 0, & \text{otherwise} \end{cases}$$



In the limit $a \to 0$ the state is localised in z_0

The Mellin transform of this function, integrating out the angular variables, gives:

$$\int |\tilde{\psi}_{z_0,a}|^2 \sin\theta \, \mathrm{d}\theta = \left[\frac{a}{4\pi^2 z_0} - \frac{a^2}{8\lambda(\pi^2 z_0^2)} + \mathcal{O}(a^3)\right]$$

This tends to a constant which vanishes as $a \to 0$. Localising in space implies delocalising in time

The series expansion for a around 0, and z_0 around ∞, are the same.

$$|\tilde{\psi}_{z_0}|^2 = \frac{\lambda}{4\pi^2 z_0} - \frac{a\lambda}{8\pi^2 z_0^2} + \frac{a^2\lambda(7-4\tau^2)}{192\pi^2 z_0^3} + \mathrm{O}\left(a^3\right)$$

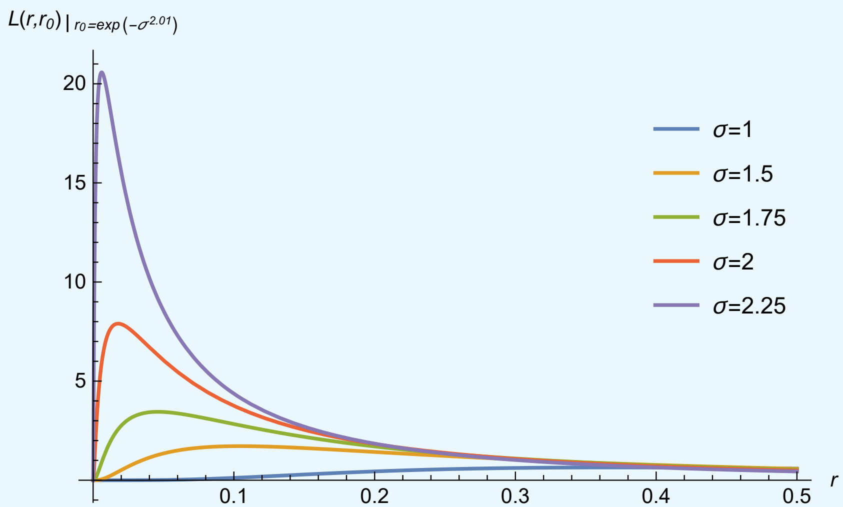
This means that a sharp localization of a particle far away from the origin implies that the particle cannot be localised in time. In accordance with the uncertainty for κ-Minkowski.

It is impossible to sharply localise a state at a point, except at the origin $x^i = 0$, which is an exceptional point.

The equivalent of the Gaussians of ordinary quantum mechanics are the log-Gaussians

$$L(r, r_0) = N \mathrm{e}^{-\frac{(\log r - \log r_0)^2}{\sigma^2}} = \mathrm{e}^{-\left(\frac{\log\left(\frac{r}{r_0}\right)}{\sigma}\right)^2} \frac{\mathrm{e}^{-\frac{9}{16}\sigma^2}}{\sqrt{\sigma}(2\pi)^{3/4}\sqrt{r_0^3}}$$

They have a maximum in $r = r_0$, which localizes at $r = r_0$ as $\sigma \to 0$, and localizes at $r = 0$ as $r_0 \to 0$, for any value of $\sigma \geq 0$.



Their Mellin transform are ordinary Gaussians (up to phases and normalizations) independent on $\boxed{r_0}$

$$\boxed{\widetilde{L}(\tau, r_0) = \frac{\sigma^{\frac{1}{2}} e^{-\frac{1}{4}\sigma^2 \tau(\tau - 3i)} r_0^{i\tau}}{2\sqrt[4]{2}\pi^{3/4}}}$$

In the double limit $r_0 \to 0$ and $\sigma \to \infty$, all $\langle r^n \rangle_L$ and all $\langle (x^0)^n \rangle_L$ go to zero as $\sigma \to \infty$.

This is a state localised both in space (at $r = 0$) and in time (at $\tau = 0$)

Localisation at arbitrary time is simply achieved multiplying the state by $\left(\frac{r}{\lambda}\right)^{\mathrm{i}\tau_0}$

With the usual abuse of notation we will call these state as $|o_\tau\rangle$.

We have argued that the origin is a special point. Does this mean that somewhere in the universe there is "the origin". A special position in space singled out by the κ-God?

Implicitly in our discussion, when we were referring to states we were assuming the existence of an observer measuring the localisation of states.

This observer is located at the origin, and he can measure with absolute precision where he is. For him "here" and "now" make sense. He cannot localise with precision states away from him, as a consequence of the noncommutativity of κ-Minkowski.

What about other observers? A different observer will be in general Poincaré transformed, i.e. translated, rotated and boosted. These operations are usually performed with an element of the Poincaré group.

One of the motivations for the introduction of κ-Minkowski is its relations to the quantum group κ-Poincaré. We should therefore used deformed transformations.

We require invariance under the transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1$

Let me give the commutations relations in a particular basis (Zakrzewski, Mercati-Sergola)

$$[a^\mu, a^\nu] = \mathrm{i}\lambda\,(\delta^\mu{}_0\, a^\nu - \delta^\nu{}_0\, a^\mu)\ , \qquad [\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma] = 0$$

$$[\Lambda^\mu{}_\nu, a^\rho] = \mathrm{i}\lambda\left[(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0)\,\Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu\right)\eta^{\mu\rho}\right].$$

With coproduct, antipode and counit

$$\Delta(a^\mu) = a^\nu \otimes \Lambda^\mu{}_\nu + 1 \otimes a^\mu\,,\ \ \Delta(\Lambda^\mu{}_\nu) = \Lambda^\mu{}_\rho \otimes \Lambda^\rho{}_\nu\,,$$

$$S(a^\mu) = -a^\nu (\Lambda^{-1})^\mu{}_\nu\,,\ \ S(\Lambda^\mu{}_\nu) = (\Lambda^{-1})^\mu{}_\nu\,, \qquad \varepsilon(a^\mu) = 0\,,\ \ \varepsilon(\Lambda^\mu{}_\nu) = \delta^\mu{}_\nu\,,$$

We represented the κ-Minkowski algebra as operators. But in doing so we had implicitly chosen an observer.

In order to take into account the fact that there are different observers we enlarge the the algebra (and consequently the space) to include the parameters of the new observers. We call then new set of states as $\mathcal{P}_\kappa$

Our (generalized) Hilbert space will now comprise not only function on space-time (either functions of r or τ), but also functions of the a's and Λ's.

We can represent the κ-Poincaré group faithfully as

$$a^\rho = -\mathrm{i}\,\frac{\lambda}{2}\left[\left(\Lambda^\mu{}_\sigma\delta^\sigma{}_0 - \delta^\mu{}_0\right)\Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu\delta^0{}_\sigma - \delta^0{}_\nu\right)\eta^{\mu\rho}\right]\Lambda^\nu{}_\alpha\frac{\partial}{\partial\omega^\mu{}_\alpha} + \mathrm{i}\frac{\lambda}{2}\left(\delta^\rho{}_0\, q^i\frac{\partial}{\partial q^i} + \delta^\mu{}_i\, q^i\right) + \frac{1}{2}\mathrm{h.c.}$$

The Λ's are represented as multiplicative operators

We have therefore that, like spacetime, the space of observers is also noncommutative, and the noncommutativity is only present in the translation sector.

We now explore the space of observers, seen as states. First consider the observer located at the origin, which is reached via the identity transformation.

Define $|o\rangle_{\mathcal{P}}$ of with the property:

$$ {}_{\mathcal{P}}\langle o|\, f(a,\Lambda)|o\rangle_{\mathcal{P}} = \varepsilon(f)\,, $$

with $f(a,\Lambda)$ a generic noncommutative function of translations and Lorentz transformation matrices, and ε the counit.

This state describes the Poincaré transformation between two coincident observers. The state is such that all combined uncertainties vanish. Coincident observers are therefore a well-defined concept in κ-Minkowski spacetime.

A change of observer will transform $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1$ and primed and unprimed coordinates correspond to different observers.

Identifying x with $\mathbb{1} \otimes x$ we generate an extended algebra $\mathcal{P} \otimes \mathcal{M}$ which extends κ-Minkowski by the κ-poincaré group algebra.

This algebra takes into account position states and observables

Remember that, just as we cannot sharply localize position states, neither we can sharply localize where the observer is.

Since Lorentz transformations commute among themselves, we can however say if two observers are just rotated with respect to each other

We can build the action of the position, translation and Lorentz transformations operator on generic functions of all those variables.

To simplify notations let us consider $1+1$ dimensions. In this case there are only two position coordinates, two translations coordinates and one Lorents transformation parametrized by ξ

The relations are $\Lambda^0{}_0 = \Lambda^1{}_1 = \cosh\xi\,,\ \Lambda^0{}_1 = \Lambda^1{}_0 = \sinh\xi\,,$

$$[a^0, a^1] = \mathrm{i}\lambda\, a^1\,, \qquad [\xi, a^0] = -\mathrm{i}\lambda \sinh\xi\,, \qquad [\xi, a^1] = \mathrm{i}\lambda\,(1-\cosh\xi)\ .$$

And the action on $\mathcal{P}$ is

$$a^0 = \mathrm{i}\lambda q \frac{\partial}{\partial q} + \mathrm{i}\lambda \sinh\xi\,\frac{\partial}{\partial\xi}\,,\ \ a^1 = q + \mathrm{i}\lambda\,(\cosh\xi - 1)\ \frac{\partial}{\partial\xi}\,,$$

States (non entangled) will be objects of the kind $|g\rangle \otimes |f\rangle$

In particular $|g\rangle \otimes |o\rangle$ is a pure translation of the state at the origin.

The new observer measures coordinates with x'. The expectation values on (normalised) transformed state is

$$\langle x'^{\mu}\rangle = \langle g| \otimes \langle o|x'^{\mu}|g\rangle \otimes |o\rangle = \langle g|\Lambda^{\mu}{}_{\nu}|g\rangle\langle o|x^{\nu}|o\rangle + \langle g|a^{\mu}|g\rangle\langle o|o\rangle \,,$$

We get:

$$\langle x'^{\mu}\rangle = \langle g|a^{\mu}|g\rangle \,,$$

The expectation value of the transformed coordinates is completely by translations. This is natural, the different observers are comparing positions, not directions.

In general

$$\langle x'^{\mu_1} \ldots x'^{\mu_n} \rangle = \langle g | a^{\mu_1} \ldots a^{\mu_n} | g \rangle \langle o | o \rangle = \langle g | a^{\mu_1} \ldots a^{\mu_n} | g \rangle \, .$$

Poincaré transforming the origin state $|o\rangle$ by a state with wavefunction $|g\rangle$ in the representation of the κ-Poincaré algebra, the resulting state will assign, to all polynomials in the transformed coordinates the same expectation value as what assigned by $|g\rangle$ to the corresponding polynomials in a^μ.

In other words, the state x'^μ is identical to the state of a^μ.

All uncertainty in the transformed coodinates $\Delta x'^\mu$ is introduced by the uncertainty in the state of the translation operator, Δa^μ.

Let me stress that, dun to the noncommutativity of translations, we do not know precisely where the new observer is unless she has just time translated the origin, i.e. $|g\rangle = |o_{a^0}\rangle_{\mathcal{P}}$.

It is also possible to see that the uncertainty of states increases with translation.

Conclusions and Outlook

There are some obvious things to do in this context. Some choices were not unique and it would be interesting to see how many of the qualitative conclusions depend on them. Some aspects are still aneddotical and can be generalized.

We considered a regime which is not very natural in physics: the effects of a quantum spacetime for which the noncommutativity parameter of space, λ is nonzero, while we ignored $\hbar$

Bringing $\hbar$ back into the picture would require us to consider momenta, and their connections with translations.

This is not easy in κ-Minkowski, where momenta form a curved space.

This led some to introduce a principle of relative locality, which reaches conclusions compatible with ours