Lecture IV

Next steps?

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Noncommutative Geometry and Applications to Quantum Physics

Euclid vs. Lorentz

As I said one of the drawbacks of the construction I am presenting is that mathematics dictates that I define spacetime as an Euclidean space, while we know that physics requires a Lorentzian signature. Several solutions have been proposed.

A fully covariant approach to spectral quantum field theory has been attempted, but the full machinery of this model does not really fit into it

Krein Spaces have been extensively used, an excellent review is the thesis of Nadir Bizi (Sorbonne), as yet unpublished

A Krein space is (very loosely speaking) a split of the Hilbert space into two subspaces, one with a positive definite linear form, the other with a negative one. Then one works separately on the two spaces. An important role is played by a “fundamental symmetry” which enables the connection with the Hilbert space

The construction has very interesting results for an older version of the model, for which the bosonic action was the curvature two form. Also in this case you reproduce the bosonic action, without the gravitational background

But the more serious drawback is that it is incompatible with the spectral action, and as such it is just a classical action. What is lacking is the boundary condition which enables the running of the constants as dictated by the renormalization group.

A Further alternative is to consider a causal structure which creates a partial ordering of states, so to reproduce the principal aspect of physics with a Lorentzian signature

In this talk I will tackle the problem using the procedure called Wick rotation.

The use of Euclidean actions if field theory is also common. What is usually said if that "in the end you Wick rotate to Lorentz signature".

Wick rotation is a procedure to change the signature of field theory. It consists (loosely speaking) in "rotating" the time derivative in the complex plane $\boxed{t \to it}$. This changes the signature of space time from a Lorentzian metric to a Euclidean one

This renders some integrals, which would be oscillatory in the functional integration, convergent since $\boxed{e^{it} \to e^{-t}}$. In some cases other regularizations work as well, and in principle they are just equivalent procedure which can work always, even if the technical difficulties can be very different

Then one Wick rotates back, i.e., undoes an operation. But in the spectral approach we cannot start unless we have an Euclidean theory. So we are not going back, we are going in unchartered territory

Usually a Wick rotation is indicated as the transformation $\boxed{t \to \mathrm{i}t}$, even if a more correct procedure would be to rotate the vierbein. Namely for each F, which depends on vierbeins

$$\boxed{\text{Wick:} \quad F\left[e^0_\mu, e^j_\mu\right] \longrightarrow \quad F\left[\mathrm{i}e^0_\mu, e^j_\mu\right], \quad j = 1,2,3.]}$$

The inverse (which is what usually people call Wick rotation) is

$$\boxed{\text{Wick}^* : \quad F\left[e^0_\mu, e^j_\mu\right] \longrightarrow \quad F\left[-\mathrm{i}e^0_\mu, e^j_\mu\right] \quad j = 1,2,3.}$$

For the bosonic part of the spectral action things go relatively without problems, the prescription is clear and the action is rotated into a new one which makes the partition function convergent

$$\boxed{\text{Wick:} \quad S^{\mathrm{E}}_{\mathrm{bos}}\left[\text{fields}, \mathrm{g}^{\mathrm{E}}_{\mu\nu}\right] \longrightarrow S^{\mathrm{E}}_{\mathrm{bos}}\left[\text{fields}, -\mathrm{g}^{\mathrm{M}}_{\mu\nu}\right] \equiv -\mathrm{i}S^{\mathrm{M}}_{\mathrm{bos}}\left[\text{fields}, \mathrm{g}^{\mathrm{M}}_{\mu\nu}\right]}$$

The fermionic sector requires some extra considerations

The group $\boxed{\mathsf{Spin}(1,3)}$ is quite different from $\boxed{\mathsf{Spin}(4)}$, $\boxed{\gamma}$ matrices, generators, charge conjugation, change. Also the fermionic action changes, since the quadratic forms have to be invariant under the proper group transformations

$$\boxed{\bar{\psi}\,\gamma^A_{\mathsf{M}}\,e^\mu_A\left(\left[\nabla^{\mathsf{LC}}_\mu\right]^{\mathsf{M}} + \mathrm{i}A_\mu\right)\psi, \quad \bar{\psi}\psi}$$

$\boxed{\bar{\psi} \equiv \psi^\dagger\gamma^0}$ and $\boxed{\nabla^{\mathsf{LC}}_\mu}$ the covariant derivative on the spinor bundle with the Levi-Civita spin-connection, which is different for Lorentzian and Euclidean

The corresponding terms with the required Spin(4) invariance are:

$$\boxed{\psi^\dagger\,\gamma^A_{\mathsf{E}}\,e^\mu_A\left[\nabla^{\mathsf{LC}}_\mu\right]^{\mathsf{E}}\psi, \quad \psi^\dagger\psi}$$

The charge conjugations are:

$$\boxed{C_{\mathsf{M}}\psi = -\mathrm{i}\gamma^{2}_{\mathsf{M}}\psi^{*} \quad ; \; C_{\mathsf{E}}\psi = \mathrm{i}\gamma^{0}_{\mathsf{E}}\gamma^{2}_{\mathsf{E}} = \widehat{C}_{\mathsf{E}}\psi^{*}}$$

The Majorana mass term is the same in both cases:

$$\boxed{\underbrace{(C_{\mathsf{E}}\psi)^{\dagger}\,\psi}_{Spin(4)\text{ inv}} = (-i\gamma^{0}_{\mathsf{E}}\gamma^{2}_{\mathsf{E}}\psi^{*})^{\dagger}\psi = \overline{(\gamma^{2}_{\mathsf{M}}\psi^{*})}\psi = -\underbrace{i\,\overline{(C_{\mathsf{M}}\psi)}\psi}_{Spin(1,3)\text{ inv}}}$$

Also the spacetime grading is the same in the two cases

$$\boxed{\gamma^{5} = \gamma^{0}_{\mathsf{E}}\gamma^{1}_{\mathsf{E}}\gamma^{2}_{\mathsf{E}}\gamma^{3}_{\mathsf{E}} = \mathrm{i}\gamma^{0}_{\mathsf{M}}\gamma^{1}_{\mathsf{M}}\gamma^{2}_{\mathsf{M}}\gamma^{3}_{\mathsf{M}}}$$

so that the definition of left and right spinor is the same

The difference between $\boxed{\psi^\dagger}$ which appers in the Euclidean, and the Lorentzian $\boxed{\bar{\psi}}$ is the presence of a $\boxed{\gamma^0}$ which must be inserted in the Lorentzian case

In NCG the fermionic spectral action is

$$\boxed{S_F = \frac{1}{2}\langle J\psi, D_A\psi\rangle}$$

Thanks to the extra degrees of freedom, the insertion of $\boxed{\gamma^0}$ by hand is not needed for this action, which therefore deals with slightly different structures.

The fermionic action is build in any case contracting the a conjugate spinor with an operator acting on a spinor. Let us look at the charge conjugation

The spacetime part of the Hilbert space splits into eigenspaces of chirality, each of which has two components, for particles and antiparticles

$$\boxed{\mathrm{Sp}(M) = H_{\mathcal{L}} \oplus H_{\mathcal{R}}}$$

with our conventions a the antiparticle of a left particle is right, and viceversa

At the same time the internal space has a similar decomposition given by the internal grading $\boxed{\gamma}$

$$\boxed{H_F = H_L \oplus H_R \oplus H_L^c \oplus H_R^c}$$

One problem with the quadruplication is the presence of "mirrors", states which have different chiralities. They have to be projected out, defining $\boxed{\mathcal{H}_+}$

$$\boxed{H_+ = (H_L)_{\mathcal{L}} \oplus (H_R)_{\mathcal{R}} \oplus (H_L^c)_{\mathcal{R}} \oplus (H_R^c)_{\mathcal{L}} = P_+ H, \quad P_+ \equiv \frac{\mathbb{I} + \Gamma}{2}}$$

This takes care of half of the extra degrees of freedom. The fermionic action is then defined as

$$\boxed{S_F = \frac{1}{2}\langle J\psi, D_A\psi\rangle \quad \psi \in H_+}$$

with $\boxed{J = C_{\mathsf{E}} \otimes J_F}$ and $\boxed{J_F = \begin{pmatrix} 0 & 0 & \mathbb{I} & 0 \\ 0 & 0 & 0 & \mathbb{I} \\ \mathbb{I} & 0 & 0 & 0 \\ 0 & \mathbb{I} & 0 & 0 \end{pmatrix} \circ cc.}$

The action reproduces correctly the Pfaffian i.e. functional integral over fermions, but this procedure only takes care of half of the extra degrees of freedom. In processes like scattering, after quatization, it is important to have the correct Hilbert space of incoming and outgoing particles.

In the bosonic spectral action the operator $\boxed{D}$ is present, not $\boxed{DP_+}$, which is not Hermitian and not a square root of the Laplacian.

Extra degrees of freedom also appear in the Euclidean quantum field theory constructed by Osterwalder and Schrader. Their construction is rendered in an axiomatic manner directly introducing the Euclidean quantum Fock space and operators acting on it,

Despite the mismatch of number of degrees of freedom per $\boxed{\vec{k}}$ the Euclidean fermionic Fock space, introduced in, does *not* contain the Lorentzian physical Fock space as a subspace (in contrast to the bosonic construction). The only connection between Lorentzian and Euclidean quantum field theories lies in the opportunity to obtain the Lorentzian Green's function via analytical continuation of matrix elements.

While Connes' spectral action approach deals with the Hilbert space of classical Euclidean fields. On the one hand, for each value $\boxed{\vec{k}}$ of the spatial momentum Lorentzian fermionic theory exhibits four one-particle states (particle and antiparticle of two polarizations). On the other hand, in the Osterwalder-Schrader's construction there are infinitely many more states: twice more polarizations, while each one particle state is also labeled by $\boxed{k_0}$, which varies continuously, so one deals with an "infiniting" rather than with a doubling. Despite some superficial similarities, the extra degrees of freedom in the two approaches are formally unrelated.

The extra degrees of freedom are taken care by the Wick rotation. It is in fact necessary to first perform the Wick rotation in order to eliminate the charge conjugation doubling

A naive attempt to remove it from the action with the $\boxed{J}$ would break the Euclidean Spin(4) symmetry.

Only the combination of Wick rotation (and identification of states described below) and the projection renders the action viable for physical applications, and free of the fermion doubling

Let us see the procedure with some more detail

First we rotate the action as in the bosonic case:

$$\boxed{\text{Wick rotation:} \quad -S_F^{\mathsf{E}}\left[\text{spinors}, e_\mu^A\right] \longrightarrow \mathrm{i}S_F^{\mathsf{M}\,\text{doubled}}\left[\text{spinors}, e_\mu^A\right]}$$

We now have a Lorentz invariant fermionic action invariant under $\boxed{Spin(1,3)}$ but still exhibiting a doubling. The spinors are in H_+, which is not anymore a Hilbert space with respect to the $Spin(1,3)$ invariant inner product

The remaining doubling consists in presence of spinors from all four subspaces of $\boxed{H_+}$: $\boxed{\left(H_L^c\right)_{\mathcal{R}}, \left(H_R^c\right)_{\mathcal{L}}, \left(H_L\right)_{\mathcal{L}}, \left(H_R\right)_{\mathcal{R}}}$

The physical Lagrangian depends on spinors just from the last two

After the Wick rotation we should perform the following identification

$$\boxed{\begin{cases} \left(\psi_L^c\right)_{\mathcal{R}} \in \underbrace{(H_L^c)_{\mathcal{R}}}_{\in H_+} \quad \text{identified with} \quad C_{\mathsf{M}}\,(\psi_L)_{\mathcal{L}}\,, \quad (\psi_L)_{\mathcal{L}} \in \underbrace{(H_L)_{\mathcal{L}}}_{\in H_+} \\ \left(\psi_R^c\right)_{\mathcal{L}} \in \underbrace{(H_R^c)_{\mathcal{L}}}_{\in H_+} \quad \text{identified with} \quad C_{\mathsf{M}}\,(\psi_R)_{\mathcal{R}}\,, \quad (\psi_R)_{\mathcal{R}} \in \underbrace{(H_R)_{\mathcal{R}}}_{\in H_+} \end{cases}.}$$

This step leads to the same formula of Barrett, who started directly Lorentzian.

We can then apply the procedure to the spectral action:

First we restore Lorentz signature in the action

$$\boxed{\begin{aligned} -S_F^{\mathsf{E}} &\to -\int d^4x\sqrt{-g^{\mathsf{M}}}\begin{bmatrix} C_{\mathsf{E}}\left(\psi_L^c\right)_{\mathcal{R}} \\ C_{\mathsf{E}}\left(\psi_R^c\right)_{\mathcal{L}} \end{bmatrix}^{\dagger}\begin{bmatrix} \mathrm{i}\not\nabla^{\mathsf{M}} & \mathrm{i}M_D \\ \mathrm{i}M_D^{\dagger} & \mathrm{i}\not\nabla^{\mathsf{M}} \end{bmatrix}\begin{bmatrix} (\psi_L)_{\mathcal{L}} \\ (\psi_R)_{\mathcal{R}} \end{bmatrix} \\ &\quad -\frac{\mathrm{i}}{2}\int d^4x\sqrt{-g^{\mathsf{M}}}\left\{\left[C_{\mathsf{E}}\,(\psi_R)_{\mathcal{R}}\right]^{\dagger} M_M\,(\psi_R)_{\mathcal{R}} + \left[C_{\mathsf{E}}\left(\psi_R^c\right)_{\mathcal{L}}\right]^{\dagger} M_M^{\dagger}\left(\psi_R^c\right)_{\mathcal{L}}\right\} \end{aligned}}$$

This action is Lorentz invariant. No modification of the inner product, like the insertion of $\boxed{\gamma^0}$, is needed.

Since $\boxed{C_{\mathsf{E}} = \mathrm{i}\gamma^0 C_{\mathsf{M}}}$ we have the manifestly Lorentz invariant action:

$$\boxed{\begin{aligned} S_F^{\mathsf{M}} &= \int d^4x\sqrt{-g^{\mathsf{M}}}\left\{\overline{(\psi_{\mathcal{L}})}\,\mathrm{i}\not\nabla^{\mathsf{M}}\psi_{\mathcal{L}} + \overline{(\psi_{\mathcal{R}})}\,\mathrm{i}\not\nabla^{\mathsf{M}}\psi_{\mathcal{R}}\right. \\ &\quad \left. - \left[\overline{(\psi_{\mathcal{L}})}\,H\,\psi_{\mathcal{R}} + \tfrac{1}{2}\overline{[C_{\mathsf{M}}\,(\psi_{\mathcal{R}})]}\,\omega\,\psi_{\mathcal{R}} + \text{c.c.}\right]\right\} \end{aligned}}$$

We still have extra degrees of freedom since each quantity which carries the index $\boxed{\text{“}c\text{”}}$ is independent from the one which does not.

It is remarkable that the path integral is not sensitive to the charge conjugation doubling, in particular the Pfaffian is reproduced correctly since

$$\boxed{\int [d\bar{\psi}][d\psi] e^{\mathrm{i}\int d^4x\, \bar{\psi}\, \mathrm{i}\not{\partial}^{\mathsf{M}}\psi} = \int [d\bar{\xi}][d\psi] e^{\mathrm{i}\int d^4x\, \bar{\xi}\, \mathrm{i}\not{\partial}^{\mathsf{M}}\psi}.}$$

The correct identification of the Hilbert space is necessary. The Lorentzian theory has to be *quantized*, and the quantum Hilbert space of asymptotic states has to be constructed. Such a space is usually referred in physical literature as a “Fock space”.

The Hamiltonian coming out of this action is not Hermitian in the Fock space. This is solved with the identification above. The rest is a straightforward exercise. In the end we obtain the correct Lorentzian signature action that you will find in textbooks.

What have we learned? I think the most intriguing element is that the Euclidean fermionic action, which uses in a crucial way the real structure of the spectral triple, and needs the fermionic quadruplication, is naturally rotated in the Lorentzian, with the elimination of the extra degrees of freedom.

There are various studies which connect spectral triples and Lorentz signatures Verch, Paschke, Sitarz, Eckstein, Franco, Besnard, Bizi, Van den Dungen, ... The considerations I exposed suggest that a possible way to obtain Lorentzian spectral triple is a rigorous treatment of Wick rotations.

Let me add a "twist" to the issue!

When I presented the Grand Symmetry I failed to stress that in order to preserve the commutativity of the algebra with its opposite algebra

$$\boxed{[a, b^0] = [a, JbJ^{-1}] = 0}$$

one has to introduce a twisted commutator, so to have a twisted spectral triple

Twisted spectral triple were introduced by Connes and Moscovici, and their are based on the introduction of a deformation of the commutator

$$\boxed{[a, b]_\rho = ab - \rho(b)a}$$

where $\boxed{\rho}$ is an automorphism of the algebra.

The Grand Symmetry requires, in the pregeometric phase, the use of twisted commutators for the algebra and the potential. In the low energy phase the automorphism reduces to the identity

One way to implement $\boxed{\rho}$ is to have a unitary operator $\boxed{R}$ so that

$$\boxed{\rho(a) = RaR^\dagger}$$

One then builds a twisted inner product, defines twisted self-adjointness and unitarity, and a twisted spectral action

$$\boxed{\langle\Psi, \Phi\rangle_\rho = \langle\Psi, R\Phi\rangle = \langle R^\dagger\Psi, \Phi\rangle.}$$

Such a $\boxed{R}$ implements naturally the split of the Hilbert space into a Krein space, if we write

$$\boxed{R = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{0} \end{pmatrix}.}$$

It would not have escaped to you attention that this is exactly the form of $\boxed{\gamma^0}$

In fact the twist needed by the grand symmetry is exactly $\boxed{\gamma^0}$, so that the twisted inner product becomes exactly the insertion of $\boxed{\gamma^0}$ necessary to make the Lorentzian inner product. Even the passage between Euclidean and Lorentzian $\boxed{\gamma}$'s is a twist with $\boxed{\gamma^0}$.

Likewise (I will omit the details), the twisted actions turn out to be the Lorentzian ones!

The picture emerging is a deep connection between the twist, Lorentzian spectral triples and Wick rotations, although unfortunately for the moment the evidence is still somewhat aneddotical

Conclusions

Noncommutative geometry starts with a view of geometry based on spectral properties, and is geared towards a profound generalization historically opened by the necessity to describe the quantum world

But then noncommutative grows to become mora a philosophy for which what is fundamental are not anymore the points, but rather the algebraic structures that we can build over them.

I tried in the last two lectures to give you the flavour of an application to the physics of fundamental interactions. What we are doing is to understand the noncommutative geometry of the standard model. This view is not the "party line" of particle physicists, but nevertheless not only gives a more general framework, which may lead to a more profound understanding, but also makes it conceivable that it may an actual contribution to phenomenology, and confront itself with experiments in a positive way.

Selected References

Apart from the ones of the previous lecture

- For the covariant approach see M. Paschke and R. Verch Local covariant quantum field theor y over spectral geometries,, Class. Quant. Grav. 21 (2004) 5299.

- A first paper on the issue of not Euclidean signatures is: A. Strohmaier On noncommutative and semi-Riemannian geometry, J. Geom. Phys. 56 (2006) 175

- And: M. Paschke and A. Sitarz Equivariant Lorentzian spectral triples, arXiv:math-ph/0611029.

- Krein spaces in this context are present in: C. Brouder, N. Bizi, F. Besnard The Standard Model as an extension of the noncommutative algebra of forms ArXiv:1504.03890

- and in: K. van den Dungen, Krein spectral triples and the fermionic action, Math. Phys. Anal. Geom. 19 (2016) no.1, 4

- Wick rotations and the spectral triple are in: F. D'Andrea, M.A. Kurkov, F. Lizzi Wick Rotation and Fermion Doubling in Noncommutative Geometry Phys.Rev. D94 (2016) 025030

Details of the Wick rotation

First we rotate the action as in the bosonic case:

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&\quad -\tfrac{\mathsf{i}}{2}\int d^4x\sqrt{-g^{\mathsf{M}}}\left\{\left[C_{\mathsf{E}}\left(\psi_R\right)_{\mathcal{R}}\right]^{\dagger} M_M \left(\psi_R\right)_{\mathcal{R}} + \left[C_{\mathsf{E}}\left(\psi_R^c\right)_{\mathcal{L}}\right]^{\dagger} M_M^{\dagger}\left(\psi_R^c\right)_{\mathcal{L}}\right\}
\end{aligned}}
$$

This action is Lorentz invariant under. No modification of the inner product, like the insertion of $\boxed{\gamma^0}$, is needed.

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&\quad \left. -\left[\overline{(\psi_{\mathcal{L}})}\,H\,\psi_{\mathcal{R}} + \tfrac{1}{2}\overline{[C_{\mathsf{M}}\,(\psi_{\mathcal{R}})]}\,\omega\,\psi_{\mathcal{R}} + \text{c.c.}\right]\right\}
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We still have extra degrees of freedom since each quantity which carries the index $\boxed{\text{“}c\text{”}}$ is independent from the one which does not.

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