Lecture III

Noncommutative geometry and “real” physics

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Noncommutative Geometry and Applications to Quantum Physics

Action and Renormalization

The remarkable fact is that the fluctuations of the Dirac operator introduced the bosonic fields, gluons which are responsible for the strong (nuclear) force, the $\boxed{W}$ and $\boxed{Z}$ bosons responsible for the weak force, the photon and another field, which in view of its coupling to the fermion is responsible for the breaking of the symmetry and to give mass to the fermions.

This is the Higgs (Englert, Brout, Guralnick, Hagen, Kibble) boson

We should get numbers. In a form which can be confronted with experiment. And while we cannot aim at agreement with 12 significant digits, at least two or three...

We have set the stage.

Ingredients:

- An algebra is a product of a continuous infinite dimensional part and a discrete finite dimensional noncommutative part. The algebra is the product $\boxed{C_0(M) \times (\mathrm{Mat}(3, \mathbb{C}) \oplus \mathbb{H} \oplus \mathbb{C})}$
- A representation of the algebra on a Hilbert space containing the known fermions. The representation is aymmetric, we can discuss the details in the exercise session.
- A generalized Dirac operator which has information on the curved background of the continuous Riemannian part as well as the masses of the fermions
- A chirality and a charge conjugation operator
- An action based on the spectrum of the operators, which we expand in series

Now we should just crank a machine

$$
\begin{gathered}
\mathcal{L}_{SM} = -\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g^2_s f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \\
\tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c^2_w} M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - igc_w(\partial_\nu Z^0_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu (W^+_\mu \partial_\nu W^-_\mu - \\
W^-_\mu \partial_\nu W^+_\mu) + Z^0_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu (W^+_\mu \partial_\nu W^-_\mu - \\
W^-_\mu \partial_\nu W^+_\mu) + A_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - \tfrac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + \\
g^2 c^2_w (Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s^2_w (A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w (A_\mu Z^0_\nu (W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \tfrac{1}{2}\partial_\mu H \partial_\mu H - 2M^2 \alpha_h H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - \tfrac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \\
\beta_h \left(\tfrac{2M^2}{g^2} + \tfrac{2M}{g} H + \tfrac{1}{2}(H^2 + \phi^0 \phi^0 + 2\phi^+ \phi^-) \right) + \tfrac{2M^4}{g^2}\alpha_h - g\alpha_h M \left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^- \right) - \\
\tfrac{1}{8}g^2 \alpha_h \left(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2 \phi^+\phi^- + 4H^2 \phi^+\phi^- + 2(\phi^0)^2 H^2 \right) - gMW^+_\mu W^-_\mu H - \\
\tfrac{1}{2}g\tfrac{M}{c^2_w} Z^0_\mu Z^0_\mu H - \tfrac{1}{2}ig \left(W^+_\mu (\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) - W^-_\mu (\phi^0 \partial_\mu \phi^+ - \phi^+ \partial_\mu \phi^0) \right) + \\
\tfrac{1}{2}g \left(W^+_\mu (H\partial_\mu \phi^- - \phi^- \partial_\mu H) + W^-_\mu (H\partial_\mu \phi^+ - \phi^+ \partial_\mu H) \right) + \tfrac{1}{2}g\tfrac{1}{c_w}(Z^0_\mu (H\partial_\mu \phi^0 - \phi^0 \partial_\mu H) + \\
M\,(\tfrac{1}{c_w} Z^0_\mu \partial_\mu \phi^0 + W^+_\mu \partial_\mu \phi^- + W^-_\mu \partial_\mu \phi^+) - ig\tfrac{s^2_w}{c_w} M Z^0_\mu (W^+_\mu \phi^- - W^-_\mu \phi^+) + igs_w M A_\mu (W^+_\mu \phi^- - W^-_\mu \phi^+) - \\
ig\tfrac{1-2c^2_w}{2c_w} Z^0_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) + igs_w A_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) - \tfrac{1}{4}g^2 W^+_\mu W^-_\mu \left(H^2 + (\phi^0)^2 + 2\phi^+\phi^- \right) - \\
\tfrac{1}{8}g^2 \tfrac{1}{c^2_w} Z^0_\mu Z^0_\mu \left(H^2 + (\phi^0)^2 + 2(2s^2_w - 1)^2 \phi^+\phi^- \right) - \tfrac{1}{2}g^2 \tfrac{s^2_w}{c_w} Z^0_\mu \phi^0 (W^+_\mu \phi^- + W^-_\mu \phi^+) - \\
\tfrac{1}{2}ig^2 \tfrac{s^2_w}{c_w} Z^0_\mu H (W^+_\mu \phi^- - W^-_\mu \phi^+) + \tfrac{1}{2}g^2 s_w A_\mu \phi^0 (W^+_\mu \phi^- + W^-_\mu \phi^+) + \tfrac{1}{2}ig^2 s_w A_\mu H (W^+_\mu \phi^- - W^-_\mu \phi^+) - \\
g^2 \tfrac{s_w}{c_w}(2c^2_w - 1) Z^0_\mu A_\mu \phi^+\phi^- - g^2 s^2_w A_\mu A_\mu \phi^+\phi^- + \tfrac{1}{2}ig_s\,\lambda^a_{ij}(\bar{q}^\sigma_i \gamma^\mu q^\sigma_j) g^a_\mu - \bar{e}^\lambda (\gamma\partial + m^\lambda_e) e^\lambda - \bar{\nu}^\lambda (\gamma\partial + \\
m^\lambda_\nu)\nu^\lambda - \bar{u}^\lambda_j (\gamma\partial + m^\lambda_u) u^\lambda_j - \bar{d}^\lambda_j (\gamma\partial + m^\lambda_d) d^\lambda_j + igs_w A_\mu \left(-(\bar{e}^\lambda \gamma^\mu e^\lambda) + \tfrac{2}{3}(\bar{u}^\lambda_j \gamma^\mu u^\lambda_j) - \tfrac{1}{3}(\bar{d}^\lambda_j \gamma^\mu d^\lambda_j) \right) + \\
\tfrac{ig}{4c_w} Z^0_\mu \{ (\bar{\nu}^\lambda \gamma^\mu (1+\gamma^5)\nu^\lambda) + (\bar{e}^\lambda \gamma^\mu (4s^2_w - 1 - \gamma^5) e^\lambda) + (\bar{d}^\lambda_j \gamma^\mu (\tfrac{4}{3}s^2_w - 1 - \gamma^5) d^\lambda_j) + (\bar{u}^\lambda_j \gamma^\mu (1 - \tfrac{8}{3}s^2_w + \\
\gamma^5) u^\lambda_j) \} + \tfrac{ig}{2\sqrt{2}} W^+_\mu \left((\bar{\nu}^\lambda \gamma^\mu (1+\gamma^5) {U^{lep}}_{\lambda\kappa} e^\kappa) + (\bar{u}^\lambda_j \gamma^\mu (1+\gamma^5) C_{\lambda\kappa} d^\kappa_j) \right) + \\
\tfrac{ig}{2\sqrt{2}} W^-_\mu \left((\bar{e}^\kappa {U^{lep}}^\dagger_{\kappa\lambda} \gamma^\mu (1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\kappa\lambda} \gamma^\mu (1+\gamma^5) u^\lambda_j) \right) + \\
\tfrac{ig}{2M\sqrt{2}} \phi^+ \left(-m^\kappa_e (\bar{\nu}^\lambda {U^{lep}}_{\lambda\kappa} (1-\gamma^5) e^\kappa) + m^\lambda_\nu (\bar{\nu}^\lambda {U^{lep}}_{\lambda\kappa} (1+\gamma^5) e^\kappa) \right) + \\
\tfrac{ig}{2M\sqrt{2}} \phi^- \left(m^\lambda_e (\bar{e}^\lambda {U^{lep}}^\dagger_{\lambda\kappa} (1+\gamma^5)\nu^\kappa) - m^\kappa_\nu (\bar{e}^\lambda {U^{lep}}^\dagger_{\lambda\kappa} (1-\gamma^5)\nu^\kappa) \right) - \tfrac{g}{2}\tfrac{m^\lambda_\nu}{M} H(\bar{\nu}^\lambda \nu^\lambda) - \tfrac{g}{2}\tfrac{m^\lambda_e}{M} H(\bar{e}^\lambda e^\lambda) + \\
\tfrac{ig}{2}\tfrac{m^\lambda_\nu}{M}\phi^0 (\bar{\nu}^\lambda \gamma^5 \nu^\lambda) - \tfrac{ig}{2}\tfrac{m^\lambda_e}{M}\phi^0 (\bar{e}^\lambda \gamma^5 e^\lambda) - \tfrac{1}{4}\,\bar{\nu}_\lambda\, M^R_{\lambda\kappa}\,(1-\gamma_5)\hat{\nu}_\kappa - \tfrac{1}{4}\,\overline{\bar{\nu}_\lambda\, M^R_{\lambda\kappa}\,(1-\gamma_5)\hat{\nu}_\kappa} + \qquad 3 \\
\tfrac{ig}{2M\sqrt{2}} \phi^+ \left(-m^\kappa_d (\bar{u}^\lambda_j C_{\lambda\kappa} (1-\gamma^5) d^\kappa_j) + m^\lambda_u (\bar{u}^\lambda_j C_{\lambda\kappa} (1+\gamma^5) d^\kappa_j) \right) + \\
\tfrac{ig}{2M\sqrt{2}} \phi^- \left(m^\lambda_d (\bar{d}^\lambda_j C^\dagger_{\lambda\kappa} (1+\gamma^5) u^\kappa_j) - m^\kappa_u (\bar{d}^\lambda_j C^\dagger_{\lambda\kappa} (1-\gamma^5) u^\kappa_j) \right) - \tfrac{g}{2}\tfrac{m^\lambda_u}{M} H(\bar{u}^\lambda_j u^\lambda_j) - \tfrac{g}{2}\tfrac{m^\lambda_d}{M} H(\bar{d}^\lambda_j d^\lambda_j) + \\
\tfrac{ig}{2}\tfrac{m^\lambda_u}{M}\phi^0 (\bar{u}^\lambda_j \gamma^5 u^\lambda_j) - \tfrac{ig}{2}\tfrac{m^\lambda_d}{M}\phi^0 (\bar{d}^\lambda_j \gamma^5 d^\lambda_j)
\end{gathered}
$$

Here the notation is as in [46], as follows.

We found the complete Lagrangian of the standard model coupled to a background gravitational field.

So what? We knew it already!

But there is an important aspect, the various constants which appear are all fixed by the entries of $\boxed{D_F}$, which are the Yukawa couplings of the fermions and their mixing.

Ex Perform a similar construction for a Grand Unified Theory

It turns out that the parameters of the Higgs are dependent on these, and are not independent, as in the standard model.

Do we have a prediction of the Higgs?

The impressive Lagrangian written above is still classical, one has to quantize, it and implement on it the renormalization programme.

As I said the Lagrangian is written coupling the SM to gravity. I will of course not quantize the gravitational field (which would give a nonrenormalizable theory) and choose as backroung Minkowsli space.

Renormalization means that all "constant" quantities in the ac- tion are a functions of the energy: running coupling constants

The running is given by the $\boxed{\beta}$ function, solution of an ordinary differential equation, calculated perturbatively to first (occasionally second, rarely third) order in $\boxed{\hbar}$ by the $\boxed{n}$-point amplitude (loop expansion)

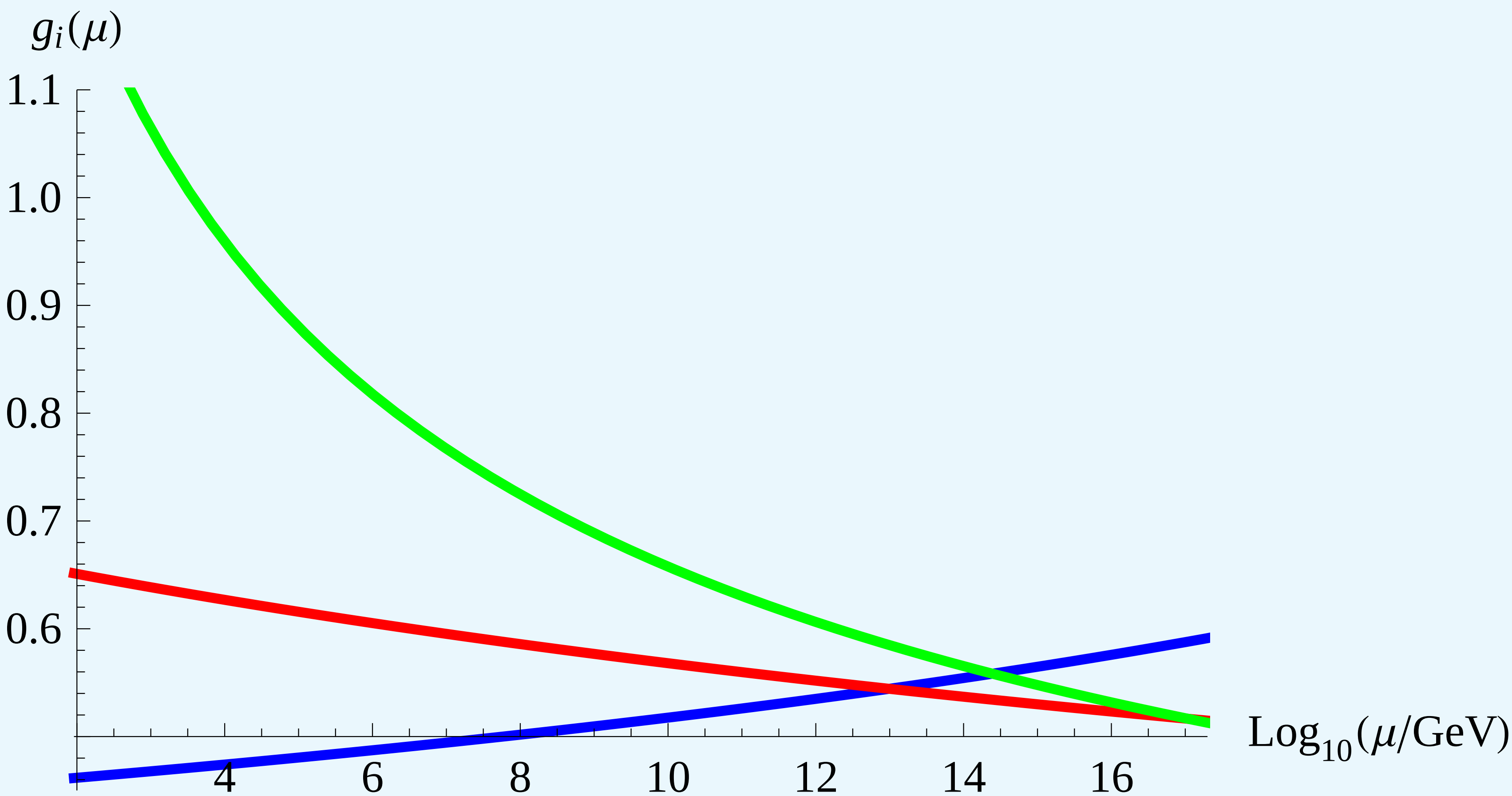
At this stage we use traditional quantum field theory techniques.

First think one has to decide is at which energy one write the big Lagrangian. This will give a boundary condition

It turns out that the fact that the top quark mass is much higher with respect to the other particles ($\boxed{\sim 170 \text{ GeV}}$ vs. $\boxed{\sim 4 \text{ GeV}}$ for the bottom) the Higgs parameters as well as the flow are dominated by this value.

One boundary condition can be given by the fact that in the model obtained cranking the machine the strength of the fundamental interaction is equal

Experimentally is known that *if there are no other particles appearing at higher energy* the three coupling constant are almost equal in one point:


$g_i(\mu)$
1.1
1.0
0.9
0.8
0.7
0.6
4
6
8
10
12
14
16
$\mathrm{Log}_{10}(\mu/\mathrm{GeV})$

As I said the Dirac operator contains all data relative to the fermions, but no information on the Higgs mass (actually vev and quartic coupling coefficient) which can be calculated from the fermion mass parameters (Yukawa couplings). These in turn are dominated by the top quark coupling.

Hence we have a "prediction" for the Higgs mass.

The prediction is $\boxed{175.1 + 5.8 - 7.2\text{GeV}}$.

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The prediction is $\boxed{175.1 + 5.8 - 7.2\text{GeV}}$.

This number is wrong.

The actual experimental value is $\boxed{125.09 \pm 32\text{GeV}}$.

Let me make a little sociological comment. In these lectures, especially this last one, I have been talking with very little mathematical rigour. This may disconcert, or even horrify mathematicians. Yet, in physics we have a very stringent rigour: experimental verification. We let ourselves do anything we want with a theory, and in the end we judge it by its predictions.

Now it depends how you consider this theory. if you take it as a mature fully formed theory then the result is wrong. That's it! Throw away the theory.

If you take it (as I do) as a tool to investigate the standard model starting from first principles, then it is remarkable that a theory based on pure mathematical result gets reasonable numbers

Alternatively take the measurement of the Higgs as a reason to understand in which direction one has to improve on the theory. And hasten to add that it possible to reconcile it with experiment.

The prediction depends on the boundary condition, and we were forced to use one for which the coupling constants of the three interactions were equal at a scale. This is true only approximately, but the picture I showed is based on a running which starts from low energy (where we have data) and extrapolates to high energy.

But if one changes the field content, then the runnings of *all* quantities change

It is known that in some supersymmetric theories the presence of the susy partners alters the running and could lead to unification.

Right handed neutrinos

Enter right handed neutrinos. The most recent addition to the particle zoo. We indirect evidence of their existence from the fact that neutrino oscillate between different flavours. This is possible only for massive particles.

We do not have a direct calculation of the mass of left handed neutrinos, but we know that it must be very small $\boxed{\lesssim 0.12 \text{ eV}}$. The following lightest is the electron at $\boxed{511000 \text{ eV}}$.

Although origin of the numerical values of all masses in the standard model is mysterious, the reason for such a low value is doubly so. And it suggests that the neutrino masses have a different origin from the other masses.

The idea is to use the so called see-saw mechanism, i.e. give a high Majorana mass to the right handed neutrino, in addition to the usual Dirac mass. Here by "high" we mean of the order of the unification energy, the scale at which the standard model would change its order parameter.

Usually one for particles one considers Dirac masses, which connect spinor with different chiralities: $\boxed{\psi_L m \psi_R}$

But also Majorana masses are possible in the Lagrangian, this connect same chirality spinors $\boxed{\psi_R m_M \psi_R}$ or $\boxed{\psi_L m'_M \psi_L}$

Therefore in general there would a mass matrix comprising both kind of masses, with the Majorana masses on the diagonal and the Dirac mass off diagonal.

In the presence of generations the matrices are not just two by two, but the idea is the same.

The eigenstate of mass are obtained diagonalizing this matrix. The see-saw mechanism assumes a matrix in which there is a Majorana mass for the right handed neutrino of the order of the unification scale, and a "normal" smaller Dirac mass. One of the eigenvalues of this matrix is $\boxed{\sim m_M}$ while the other is $\boxed{\frac{m}{m_M^2} \ll m}$

Although as we said the constraints given by NCG are quite stringent, it turns out that the slot corresponding to a Majorana mass matrix for right handed neutrino is allowed.

This means that now we have another large scale in the model, and this may change the running of the $\boxed{beta}$ functions, and cause on one side the unification of the three interactions, and on the other a different value for the Higgs mass.

A different $\boxed{D_F}$ should also cause different one-forms, i.e. different bosonic fields.

But an explicit calculation shows that the extra term in $\boxed{D_F}$ commutes with the algebra, and therefore no extra boson, no different value for the Higgs.

These considerations however suggested to Chamseddine and Connes to consider the entry in $\boxed{D_F}$ to be an independent field.

This extra field has some of the properties of the Higgs, and it couples to it, changing the running.

Such a field had already appeared in the literature when it was realised that a relatively light Higgs at $\boxed{\text{125 GeV}}$ creates a dangerous instability. At some energy, intermediate between the present scale and the unification scale, the coefficient of the quartic term in the Higgs potential becomes negative, turning the Mexican hat into a potential not bounded from below.

The problem is that in NCG all bosons (so far are in one-forms which are obtained by commuting $\boxed{D}$ with elements of the algebra. In this way we obtained $\boxed{W}$, $\boxed{Z}$, photons gluons and the Higgs. Putting this field by hand is at best unpleasant.

Moreover the addition of an extra boson, with its mass and coupling parameter lowers the predictive power of the model, with the addition of the extra parameter one can only state that the model is compatible with data.

We should probably ask more to the model! And go back to its roots.

Since the conditions for a spectral triple to describe a manifold have been cast algebraically, we can see which noncommutative finite dimensional $\boxed{C^*}$ algebras satisfy the conditions. And I remind you that a finite dimensional $\boxed{C^*}$ is necessarily a sum of matrices over the reals, complex or quaternions

This is a straightforward exercise you. But you need use all of the five elements of the triple. The result is that the finite part of the spectral triple must have a well defined form:

$$\boxed{\mathbb{M}(\mathbb{H})_a \oplus \mathbb{M}(\mathbb{C})_{2a}}$$

for $\boxed{a}$ integer.

The direct sum of matrices of quaternions (which in turn can be represented as $\boxed{2 \times 2}$ matrices) and matrices of complex number of the same size.

We need the algebra to be represented on an Hilbert space of dimension $\boxed{n = 2(2a)^2}$ (up to generation replicas)

The gauge group of this algebra is made of the unitary operators, and the symmetry will be “broken”, thus reducing the gauge group.

Hence there is not much freedom in the game, since you need to see if there is way to obtain the algebra of the standard model

This algebra acts on a finite Hilbert space of dimension $\boxed{2(2a)^2}$.
For a non trivial grading it must be $\boxed{a \geq 2}$

$$\boxed{\mathcal{A}_F = \mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C})}$$

Hence an Hilbert space of dimension $\boxed{2(2 \cdot 2)^2 = 32}$, the dimension of $\boxed{\mathcal{H}_F}$ for one generation.

The grading condition $\boxed{[a, \Gamma] = 0}$ reduces the algebra to the left-right algebra

$$\boxed{\mathcal{A}_{LR} = \mathbb{H}_L \oplus \mathbb{H}_R \oplus \mathbb{M}_4(\mathbb{C})}$$

The order one condition reduces further the algebra to $\mathcal{A}_{sm}$, i.e. the algebra whose unimodular group is U(1)×SU(2)×U(3)

The group $\boxed{SU(4) \times SU(2) \times SU(2)}$ has been introduced long ago (Pati-Salam). Is one of the first example of a Grand Unified Theory. There is a sort of fourth colour (lepton number) and is left-right symmetric.

This unified theory should break to the standard model. A field called $\boxed{\sigma}$ (analog to the Higgs) is necessary. This field appears in $\boxed{D}$ in the position corresponding to a particular form of the neutrino mass (Majorana). It turns out that precisely in that spot (and not many others) it is possible to put a nonzero value!

But cranking of the machine does not produce a contribution to the one form, the extra term commutes with the algebra. Hence must included it by hand. Which is unpleasant

Doing again the running of the physical quantitates with this field does change the Higgs mass, making it compatible with the experimental value

Physics is therefore telling us that into his framework right handed neutrinos, and Majorana masses are crucial

Can we avoid adding this field by hand? There are three possible solutions

- Enlarge the Hilbert space introducing new fermions and new interactions. Stephan

- Consider a Grand Symmetry based on $\boxed{\mathbb{M}(\mathbb{H})_4 \oplus \mathbb{M}(\mathbb{C})_8}$ Devastato FL Martinetti

- Violate one of the conditions (order one) Chamseddine, Connes Van Suijlekom

The latter solutions allow the introduction of a new field $\boxed{\sigma}$ which not only fixes the mass of the Higges making it compatible with 126 GeV, but also solves the possible instability of the theory.

A. Devastato, P. Martinetti and myself proposed a solution: a grand symmetry.

In NCG the usual grand unified groups, such as $\boxed{SU(5)}$ or $\boxed{SO(10)}$ do not work. There are very few representations of associative algebras, as opposed to groups. Finite dimensional algebras only have one nontrivial IRR

Fortunately in the standard model there are only weak doublets and colour triplets, so it works

Recall that a finite "manifold" is an algebra: $\boxed{M_a(\mathbb{H}) \oplus M_{2a}(\mathbb{C})}$ acting on a $\boxed{2(2a)^2}$ dimensional Hilbert space. So far we had $\boxed{a = 2,\ 2(2a)^2 = 32 \times 3 = 96}$

The numerology comes out correct

For $\boxed{M_4(\mathbb{H}) \oplus M_8(\mathbb{C})}$ one requires a $\boxed{2(2 \cdot 4)^2 = 128}$ dimensional space. (384 taking generations into account)

This is exactly the dimension of the Hilbert space if we take the fermion doubling into account. This overcounting had been perceived as a nuisance if not a problem. One had to project states out, and the unphysical redundancy was unexplained

It is necessary to look at Hilbert space with different eyes

$$\boxed{\mathcal{H} = sp(L^2(M)) \otimes \mathcal{H}_F = L^2(M) \otimes \mathsf{H}_F}$$

where now the dimensions of $\boxed{\mathsf{H}_F}$ is 384

It is still possible to represent the gran algebra $\boxed{M_4(\mathbb{H}) \oplus M_8(\mathbb{C})}$ satisfying all of the manifold conditions. This is highly nontrivial if one keeps the same Hilbert space.

But this time the algebra does not act diagonally on the spinor indices. it mixes them. In addition one has to consider a particular form of spectral triple: twisted triples, for which the commutators present in the various conditions are twisted by an isomorphism of the algebra.

If I have no time I will not write explicitly the details of the representation (on particle and anti particles) because they are rather involved. The key point is that in the process spacetime indices, related to the Euclidean symmetries, mix with internal, gauge indices.

We envisage this Grand Symmetry to belong to a pre geometric phase. At this stage all elements of $\boxed{D_F}$ may be negligible, and the spinor part of the direct operator $\boxed{\partial\!\!\!/}$ will cause the “breaking” to a phase in which the symmetries of the phase space emerge

In particular, the order one condition for $\boxed{\partial\!\!\!/}$ causes the reduction of the algebra to $\boxed{\mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C})}$

And there is an added bonus:

This grand algebra, and a corresponding $\boxed{D}$ operator, have "more room" to operate. Although the Hilbert space is the same, the fact that we abandoned the factorization of the internal indices, gives us more entries to accommodate the Majorana masses

Hence we can put a Majorana mass for the neutrino and at the same time satisfy the order one condition. Then the one form corresponding to this $\boxed{D_\nu}$ will give us the by now famous field $\boxed{\sigma}$, which can only appear before the transition to the geometric spacetime

The natural scale for this mass is to be above a transition which gives the geometric structure. Therefore it is natural that it may be at a high scale. How high we can discuss

The grand symmetry is no ordinary gauge symmetry, there is never a $\boxed{SU(8)}$ in the game for example

It represents a phase in which the internal noncommutative geometry contains also the spin structure, even the Lorentz (Euclidean) structure of space time in a mixed way

The differentiation between the spin structure of spacetime, and the internal gauge theory comes as a breaking of the symmetry, triggered by $\boxed{\sigma}$, which now appears naturally has having to do with the geometry of spacetime.

The fermion doubling was not a problem after all. . .

Selected References

Apart from the ones of the previous lecture

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- A lot of preparatory work was done by the Marseille group, notably Daniel Kastler, Bruno Iochum and Thomas Schuker and several young people, in a series of papers lasting over a decade. No review would be complete without mentioning their work. One of the latest review by T. Schucker which still has some actuality is Forces from Connes' geometry Lect.Notes Phys. 659 (2005) 285-350.

- The final cranking is in the paper of Chamseddine, Connes and Marcolli Gravity and the standard model with neutrino mixing Adv.Theor.Math.Phys. 11 (2007) no.6, 991

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- The grand symmetry first appeared in: Devastato, FL, Martinetti Grand Symmetry, Spectral Action, and the Higgs mass JHEP 1401 (2014) 042

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Details of the representation

The Hilbert spacehas 384 finite degrees of freedom, four are coming from the spinorial part and 96 from the internal part $\mathbf{H}_F$. This figure is the product of the eight particles (neutrino, electron, up and down quarks in three different coulours), times two chiralities, with theor respective antiparticles times three generations. The 384 includes the overcounting due to the fermion doubling

It is useful to arrange $\mathbf{\Psi} \in \mathbf{H}$ into a nested set of matrices. We start by expressing $\mathbf{\Psi}$ as a rectangular block matrix:

$$\mathbf{\Psi} = \begin{pmatrix} \mathbb{P} \\ \mathbb{A} \end{pmatrix}$$

where $\mathbb{P}$ and $\mathbb{A}$ stand for "particle" and "antiparticle" respectively. In other words we are splitting $\mathbf{H}$ into its particle and antiparticle subspaces:

$$\mathbf{H} = \begin{pmatrix} \mathbb{H} \\ \mathbb{H}^c \end{pmatrix}$$

We then express $\mathbb{P}$ and $\mathbb{A}$ as 4×4 block matrices:

$$\mathbb{P} = \begin{pmatrix} \mathsf{V}_R & \mathsf{U}^1_R & \mathsf{U}^2_R & \mathsf{U}^2_R \\ \mathsf{E}_R & \mathsf{D}^1_R & \mathsf{D}^2_R & \mathsf{D}^2_R \\ \mathsf{V}_L & \mathsf{U}^1_L & \mathsf{U}^2_L & \mathsf{U}^2_L \\ \mathsf{E}_L & \mathsf{D}^1_L & \mathsf{D}^2_L & \mathsf{D}^2_L \end{pmatrix} \;;\; \mathbb{A} = \begin{pmatrix} \mathsf{V}^c_R & \mathsf{E}^c_R & \mathsf{V}^c_L & \mathsf{V}^c_L \\ \mathsf{U}^{1c}_R & \mathsf{D}^{1c}_R & \mathsf{U}^{1c}_L & \mathsf{D}^{1c}_L \\ \mathsf{U}^{2c}_R & \mathsf{D}^{2c}_R & \mathsf{U}^{2c}_L & \mathsf{D}^{2c}_L \\ \mathsf{U}^{3c}_R & \mathsf{D}^{3c}_R & \mathsf{U}^{3c}_L & \mathsf{D}^{3c}_L \end{pmatrix}$$

where the subscript $1, 2, 3$ represent colour; the subscripts L and R represent the chiral grading in the internal space, remember that the antiparticle (identified by the superscript c) of a left handed state is right handed and viceversa; $\mathsf{V}, \mathsf{E}, \mathsf{U}, \mathsf{D}$ are columns whose elements the fermions: neutrino, electron, up and down quark respectively, for the first elements, repeated for the three generations, i.e.:

$$\mathsf{V} = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \; ; \mathsf{E} = \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} \; ; \; \mathsf{U} = \begin{pmatrix} u \\ c \\ t \end{pmatrix} \; ; \; \mathsf{D} = \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

The subscripts L and R refer to the index in the internal space. Each of the elements of the latter vectors are Dirac fermions, therefore there should be a further spinorial index, which for the moment we omit.

To isolate quarks and leptons useful to define the projection matrix $\mathbb{e}_{11}$ as

$$\mathbb{e}_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so that the leptonic and hadronic part of the Hilbert space are defined as:

$$\mathbf{H}_l = \begin{pmatrix} \mathbb{0} & \\ & \mathbb{e}_{11} \end{pmatrix} \mathbf{H} + \begin{pmatrix} \mathbb{1} & \\ & \mathbb{0} \end{pmatrix} \mathbf{H}\,\mathbb{e}_{11} \; ; \; \mathbf{H}_q = \begin{pmatrix} \mathbb{1} & \\ & \mathbb{1} - \mathbb{e}_{11} \end{pmatrix} \mathbf{H} + \begin{pmatrix} \mathbb{0} & \\ & \mathbb{1} \end{pmatrix} \mathbf{H}\,(\mathbb{1} - \mathbb{e}_{11})$$

A generic endomorphism of $\mathbf{H}_F$ can be represented by the sum of left actions of 8×8 matrices on the left times 4×4 matrices on the right, and the matrix multiplication is of course the usual row by column. An alternative representation, the "opposite" representation would be obtained multiplying colum by row a

4×4 matrix on the left times a 8×8 matrix on the right. Discuss the dimensions.

The element $\mathbf{a} = (c, q, m) \in \mathcal{A}$ acts on $\mathbf{H}$ from the left as follows as follows. Define two 4×4 block matrices $\mathbb{a}_u$ and $\mathbb{a}_l$, where l and u stand for upper and lower, as

$$\mathbb{a}_u = \left(\begin{array}{cc|c} c & 0 & \\ 0 & \bar{c} & \\ \hline & & q \\ & & \end{array}\right) \quad , \quad \mathbb{a}_l = \left(\begin{array}{c|c} c & \\ \hline & \\ & m \\ & \end{array}\right)$$

The algebra is then represented as an endomorphism as*

$$\mathbf{a\Psi} = \begin{pmatrix} \mathbb{a}_u \mathbb{P} \\ \mathbb{a}_l \mathbb{A} \end{pmatrix}$$

The charge conjugation operator $\mathbf{J}_F$ acts exchanging particles with antiparticles as follows†:

$$\mathbf{J}_F \mathbf{\Psi} = \begin{pmatrix} \mathbb{A}^\dagger \\ \mathbb{P}^\dagger \end{pmatrix}$$

The action of the opposite algebra can be represented as the

*A generic automorphism of $\mathbf{H}$ is given by the left action of an 8×8 matrix, and the right action of a 4×4 matrix. for precision we should have written $\mathbf{a\Psi 1}_4$, with an identity matrix on the right. To ease notations we will omit the identity matrices when no confusion may occur.

†Although $\mathbb{J}_F$ is an antilinear operator we still use the matrix multiplication notation.

action:

$$\mathbf{a}^o\mathbf{\Psi} = \mathbf{J}_F\bar{\mathbf{a}}\mathbf{J}_F\mathbf{\Psi} = \mathbf{\Psi}\circ\bar{\mathbf{a}}$$

where by $\circ$ we indicate the action of the opposite algebra which acts by "colums by row" multiplication and the overbar is complex conjugation.

It is easy to verify that for each pair of elements $\mathbf{a}, \mathbf{b} \in \mathcal{A}_F$

$$\mathbf{a}\mathbf{J}_F\mathbf{b}\mathbf{J}_F\mathbf{\Psi} = \mathbf{J}_F\mathbf{b}\mathbf{J}_F\mathbf{a}\mathbf{\Psi} = \mathbf{a}\mathbf{\Psi}\circ\bar{\mathbf{b}}$$

i.e. the order 0 condition is satisfied. The operator $\mathbf{J}_F$ enables the definition of the action of the unimodular (unitary of unit determinant) elements of the algebra on the Hilbert space and thus define the gauge group. These elements are defined from the unitary elements of $\mathcal{A}_F$ as $\mathbf{u}\mathbf{J}_F\mathbf{u}^\dagger\mathbf{J}_F$. Hence the action of the unimodular elements of the algebra gives the gauge group, for $g = \{\lambda, W, G\} \in U(1) \oplus SU(2) \oplus SU(3)$ we have:

$$\mathbf{g\Psi} = \left(\begin{array}{c|c} \begin{array}{cc|c} \lambda^3 & 0 & \\ 0 & \bar\lambda^3 & \\ \hline & & W \end{array} & \\ \hline & \begin{array}{c|c} \lambda^3 & \\ \hline & \bar\lambda G \end{array} \end{array}\right) \left(\begin{array}{c} \mathbb{P} \\ \hline \mathbb{A} \end{array}\right) \circ \left(\begin{array}{c|c} \begin{array}{c|c} \bar\lambda^3 & \\ \hline & \lambda G^\dagger \end{array} & \\ \hline & \begin{array}{cc} \bar\lambda^3 & 0 \\ 0 & \lambda \end{array} \end{array}\right.$$

the action of $U(1)$ on the various particles is according to the representation $\mathrm{e}^{Y\lambda}$ given by the usual hypercharge assignments:

	$\mathsf{V}_L, \mathsf{E}_L$	$\mathsf{U}^i_L, \mathsf{D}^i_L$	V_R	E_R	U^i_R	D^i_R
Y	-1	$\frac{1}{3}$	0	-2	$\frac{4}{3}$	$-\frac{2}{3}$

with the antiparticle having the opposite hypercharge of the corresponding particles.

A generic Dirac operator which satisfies the order one condition and commutes with $\mathbf{J}_F$, must be the sum of three Hermitean operators:

$$\mathbf{D}_F = \mathbf{D}_C + \mathbf{J}_F \mathbf{D}_C \mathbf{J}_F + \mathbf{D}_M$$

where $\mathbf{D}_C$ commutes with any element of the opposite algebra $[\mathbf{D}_C, \mathbf{J}_F \mathbf{a} \mathbf{J}_F] = 0$, and has zero componets in the directions $\mathbb{e}_{15} \otimes \mathbb{e}_{11}$ and $\mathbb{e}_{51} \otimes \mathbb{e}_{11}$, while $\mathbb{D}_M$ commutes with both the algebra $\mathcal{A}$ and its opposite $\mathcal{A}^o$. These conditions limit severely the form of $\mathbf{D}_C$ and $\mathbf{D}_M$. To find their form we need the explicit form of $(\mathcal{A}^o)'$, the commutant of the opposite algebra $\mathcal{A}^o$. Explicit computation shows that an element of it, $\mathbf{a}'$, is given by three complex numbers f, g, h, a 2×2 matrix r, and a 4×4 matrix $\mathbb{m}$ and acts on $\mathbf{\Psi}$ as

$$\mathbf{a}'\mathbf{\Psi} = \left(\begin{array}{c|c|c|c|c} t_{11} & & & t_{12} & \\ \hline & f & & & \\ \hline & & g1_2 & & \\ \hline t_{21} & & & t_{22} & \\ \hline & & & & h1_3 \end{array}\right) \left(\begin{array}{c} \mathbb{P} \\ \hline \mathbb{A} \end{array}\right) \mathbb{m}$$

$\mathbf{D}_C$ must commute with any matrix of the form above it must have a definite form. Which happens to have all of the known Yukawa couplings, plus some extra slots which we will not discuss.

The 15 and 51 slots are taken care by $\mathbf{D}_M$ which is

$$\Upsilon \mathbb{e}_{15} \otimes \mathbb{e}_{11} + \overline{\Upsilon} \mathbb{e}_{51} \otimes \mathbb{e}_{11}$$

The action of $e_{15} \otimes \mathbb{e}_{11}$ and $e_{51} \otimes \mathbb{e}_{11}$ singles out the right handed neutrino states V_R in the Hilbert space of states. Since $\mathbf{D}_M$ commutes with the algebra and its opposite it does not contribute to the potential one forms. That is, in the bosonic action it will not give rise to a field.

Let us first consider the operator $\mathbf{D}_{0F}$, it is a 96$\otimes$96 matrix which acts trivially on the spinor indices and, unlike the representation of the algebra, is not diagonal in the matrix indices. It contains the Yukawa couplings of the fermions, which give rise to two kind of masses, Dirac and Majorana.

Since Dirac and Majorana masses play a slightly different role we will split D_0F into two parts:

$$D_{0F} = D_{0D} + D_{0M}$$

Let us consider first the Dirac mass term matrix, which we split in its action on leptons and quarks

$$\begin{aligned} D_{0D}[\Psi] &= D_l[\Psi] + D_q[\Psi] \\ &= \begin{pmatrix} \mathbb{Y}_{0l} & \\ & \mathbb{e}_{11} \end{pmatrix} \begin{pmatrix} \mathbb{P} & \mathbb{0} \\ \mathbb{0} & \mathbb{A} \end{pmatrix} \begin{pmatrix} \mathbb{e}_{11} & \\ & \mathbb{Y}_{0l} \end{pmatrix} + \begin{pmatrix} \mathbb{Y}_{0q} & \\ & \mathbb{1} - \mathbb{e}_{11} \end{pmatrix} \begin{pmatrix} \mathbb{P} & \mathbb{0} \\ \mathbb{0} & \mathbb{A} \end{pmatrix} \Bigg(\mathbb{1} - \mathbb{e}_1 \end{aligned}$$

with the same dimensional blocks as (), the block is turn are split as

$$\mathbb{Y}_{0l} = \begin{pmatrix} & & \mathsf{Y}_\mathsf{V}^\dagger & \\ & & & \mathsf{Y}_\mathsf{E}^\dagger \\ \mathsf{Y}_\mathsf{E} & & & \\ & \mathsf{Y}_\mathsf{V} & & \end{pmatrix} \; ; \; \mathbb{Y}_{0q} = \begin{pmatrix} & & \mathsf{Y}_\mathsf{U}^\dagger & \\ & & & \mathsf{Y}_\mathsf{D}^\dagger \\ \mathsf{Y}_\mathsf{D} & & & \\ & \mathsf{Y}_\mathsf{U} & & \end{pmatrix}$$

Majorana mass terms instead couples R neutrinos with R neutrinos:

$$D_{0M}[\mathbf{\Psi}] = \begin{pmatrix} & \mathbb{e}_{11} \\ \mathbb{Y}_{0M} & \end{pmatrix} \begin{pmatrix} \mathbb{P} & \mathbb{0} \\ \mathbb{0} & \mathbb{A} \end{pmatrix} \begin{pmatrix} & \mathbb{Y}_{0M}^{\dagger} \\ \mathbb{e}_{11} & \end{pmatrix}$$

and $\mathbb{Y}_{0M}$ is the diagonal matrix

$$\mathbb{Y}_{0M} = \begin{pmatrix} \mathsf{Y}_M & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

The covariant Dirac operator gives the presence of a Higgs field acting on the doublets

$$\mathbb{Y}_l = \left(\begin{array}{c|c} 0 & \begin{pmatrix} \mathsf{Y}_\mathsf{V}^\dagger & 0 \\ 0 & \mathsf{Y}_\mathsf{E}^\dagger \end{pmatrix} \otimes H \\ \hline \begin{pmatrix} \mathsf{Y}_\mathsf{V} & 0 \\ 0 & \mathsf{Y}_\mathsf{E} \end{pmatrix} \otimes H^\dagger & 0 \end{array}\right)$$

$$\mathbb{Y}_q = \left(\begin{array}{c|c} 0 & \begin{pmatrix} \mathsf{Y}_\mathsf{U}^\dagger & 0 \\ 0 & \mathsf{Y}_\mathsf{D}^\dagger \end{pmatrix} \otimes H \\ \hline \begin{pmatrix} \mathsf{Y}_\mathsf{U} & 0 \\ 0 & \mathsf{Y}_\mathsf{D} \end{pmatrix} \otimes H^\dagger & 0 \end{array}\right)$$

It is easy to see by direct calculation from the explicit form of the element $\mathbf{a}$ that instead

$$[\mathbf{D}_{0M}, \mathbf{a}] = 0 \Rightarrow \mathbf{D}_{0M} = \mathbf{D}_M$$

and therefore the Majorana mass term does not give rise to an extra field.

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Details of the representation of the Grand Algebra

There are several finite dimensional algebras in this game, and I want to look at their representations

Ultimately we want to ge to the the standard model algebra

$$\boxed{\mathcal{A}_{sm} = \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{M}_3(\mathbb{C}),}$$

$\boxed{\mathbb{H}}$ are the quaternions, which we represent as $\boxed{2 \times 2}$ matrices

It is possible to have this emerge from the most general algebra which satisfies the condition of being a noncommutative manifold

$$\boxed{\mathcal{A}_{\mathcal{F}} = \mathbb{M}_a(\mathbb{H}) \oplus \mathbb{M}_{2a}(\mathbb{C}) \qquad a \in \mathbb{N}.}$$

This algebra acts on a finite Hilbert space of dimension $\boxed{2(2a)^2}$. For a non trivial grading it must be $\boxed{a \leq 2}$

$$\boxed{\mathcal{A}_F = \mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C})}$$

Hence an Hilbert space of dimension $\boxed{2(2 \cdot 2)^2 = 32}$, the dimension of $\boxed{\mathcal{H}_F}$ for one generation.

The grading condition $\boxed{[a, \Gamma] = 0}$ reduces the algebra to the left-right algebra

$$\boxed{\mathcal{A}_{LR} = \mathbb{H}_L \oplus \mathbb{H}_R \oplus \mathbb{M}_4(\mathbb{C})}$$

The order one condition reduces further the algebra to $\mathcal{A}_{sm}$, i.e. the algebra whose unimodular group is U(1)×SU(2)×U(3)

Let us look in detail to a vector in the Hilbert space:

$$\boxed{\Psi^{\mathsf{C}\mathrm{I}m}_{s\dot{s}\alpha}(x) \in \mathcal{H} = L^2(\mathcal{M}) \otimes \mathsf{H}_F = sp(L^2(\mathcal{M})) \otimes \mathcal{H}_F}$$

Note the difference between $\boxed{\mathcal{H}_F}$, which is 96 dimensional, and $\boxed{\mathsf{H}_F}$ which is 384 dimensional. The meaning of the indices is as follows:

$$\boxed{\Psi^{\mathsf{C}\mathrm{I}m}_{s\dot{s}\alpha}(x)}$$

$\boxed{\begin{array}{r} s = r, l \\ \dot{s} = \dot{0}, \dot{1} \end{array}}$ are the spinor indices. They are not internal indices in the sense that the algebra $\mathcal{A}_F$ acts diagonally on it. They take two values each, and together they make the four indices on an ordinary Dirac spinor.

$$\boxed{\Psi^{\mathsf{c}\mathrm{I}m}_{s\dot{s}\alpha}(x)}$$

$\boxed{\mathrm{I} = 0, \ldots 3}$ indicates a "lepto-colour" index. The zeroth "colour" actually identifies leptons while $\boxed{\mathrm{I} = 1, 2, 3}$ are the usual three colours of QCD.

$$\boxed{\Psi^{\mathsf{c}\mathrm{I}m}_{s\dot{s}\alpha}(x)}$$

$\boxed{\alpha = 1 \ldots 4}$ is the flavour index. It runs over the set $\boxed{u_R, d_R, u_L, d_L}$ when $\boxed{\mathrm{I} = 1, 2, 3}$, and $\boxed{\nu_R, e_R, \nu_L, e_L}$ when $\boxed{\mathrm{I} = 0}$. It repeats in the obvious way for the other generations.

$$\boxed{\Psi^{\mathsf{C}\mathrm{I}m}_{s\dot{s}\alpha}(x)}$$

$\boxed{\mathsf{C} = 0, 1}$ indicates whether we are considering "particles" ($\boxed{\mathsf{C} = 0}$) or "antiparticles" ($\boxed{\mathsf{C} = 1}$).

$$\boxed{\Psi^{\mathsf{C}\mathrm{I}m}_{s\dot{s}\alpha}(x)}$$

$\boxed{m = 1, 2, 3}$ is the generation index. The representation of the algebra of the standard model is diagonal in these indices, the Dirac operator is not, due to Cabibbo-Kobayashi-Maskawa mixing parameters. For this seminar plays no role, and will ignored.

We can now give explicitly the algebra representations in term of these indices.

We start from $\boxed{\mathcal{A}_F = \mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C})}$, a generic element will depend on $\boxed{4 \times 4}$ complex matrix $\boxed{m}$, and a $\boxed{2 \times 2}$ matrix of quaternions $\boxed{q}$, which we may also see as a $\boxed{4 \times 4}$ with some conditions

The representation in its fullness is

$$\boxed{A_{s\dot{s}\mathsf{DJ}\alpha}^{t\dot{t}\,\mathsf{CI}\beta} = \delta_s^t \delta_{\dot{s}}^{\dot{t}} \left(\delta_0^{\mathsf{C}} \delta_{\mathsf{J}}^{\mathsf{I}} Q_\alpha^\beta + \delta_1^{\mathsf{C}} M_{\mathsf{J}}^{\mathsf{I}} \delta_\alpha^\beta \right)}$$

Note the two $\boxed{\delta}$'s at the beginning which show that the algebra acts trivially on the spacetime indices, and the fact that the two matrices act on different indices. This ensures the order zero condition, namely exchanging particles with antiparticles, the job done by $\boxed{J}$, the two representations commute.

The representations of the other algebra are similar, in the case of the standard model there is a differentiation with the leptocolour indices.

The order one condition and a $\boxed{\nu}$ Majorana mass cause the reduction to $\boxed{C^\infty(\mathcal{M}) \otimes \mathcal{A}_{sm}}$, represented as

$$\boxed{a = \{m, q, c\} \text{ with } m \in C^\infty(\mathcal{M}) \otimes \mathbb{M}_3(\mathbb{C}),\ q \in C^\infty(\mathcal{M}) \otimes \mathbb{H},\ c \in C^\infty(\mathcal{M}) \otimes \mathbb{C}}$$

is

$$\boxed{a^{t\dot t \mathsf{CI}\beta}_{s\dot s \mathsf{DJ}\alpha} = \delta^t_s \delta^{\dot t}_{\dot s} \left(\delta^{\mathsf{C}}_0 \delta^{\mathsf{I}}_{\mathsf{J}} \left(\mathsf{q}^\beta_\alpha + \mathsf{c}^\beta_\alpha \right) + \delta^{\mathsf{C}}_1 \left(\mathsf{m}^{\mathsf{I}}_{\mathsf{J}} + \tilde{\mathsf{c}}^{\mathsf{I}}_{\mathsf{J}} \right) \delta^\beta_\alpha \right)}$$

where we use the following $\boxed{4 \times 4}$ complex matrices:

$$\mathsf{q} = \begin{pmatrix} 0_2 & \\ & q \end{pmatrix}_{\alpha\beta}, \qquad \mathsf{c} = \begin{pmatrix} c & & \\ & \bar{c} & \\ & & 0_2 \end{pmatrix}_{\alpha\beta}, \qquad \tilde{\mathsf{c}} = \begin{pmatrix} c & \\ & 0_3 \end{pmatrix}_{\mathsf{IJ}}, \qquad \mathsf{m} = \begin{pmatrix} 0 & \\ & m \end{pmatrix}_{\mathsf{IJ}}$$

The breaking $\boxed{\mathcal{A}_F}$ to $\boxed{\mathcal{A}_{sm}}$ goes with the chirality and first order conditions

I can similarly write down the Dirac operator

$$\boxed{D = \partial\!\!\!/ \otimes \mathbb{I}_{96} + \gamma^5 \otimes D_F}$$

$$\boxed{D_F = \begin{pmatrix} 0_{8N} & \mathcal{M} & \mathcal{M}_R & 0_{8N} \\ \mathcal{M}^\dagger & 0_{8N} & 0_{8N} & 0_{8N} \\ \mathcal{M}_R^\dagger & 0_{8N} & 0_{8N} & \bar{\mathcal{M}} \\ 0_{8N} & 0_{8N} & \mathcal{M}^T & 0_{8N} \end{pmatrix}.}$$

$\boxed{\mathcal{M}}$ contains the Dirac-Yukawa couplings. It links left with right particles. $\boxed{\mathcal{M}_R = \mathcal{M}_R^T}$ contains Majorana masses and links righr particles with right antiparticles. $\boxed{\mathcal{M} = \begin{pmatrix} M_u & 0_{4N} \\ 0_{4N} & M_d \end{pmatrix} \qquad \mathcal{M}_R = \begin{pmatrix} M_R & 0_{4N} \\ 0_{4N} & 0_{4N} \end{pmatrix}}$ where $\boxed{M_u}$ containins the masses of the up, charm and top quarks and the neutrinos (Dirac mass), $\boxed{M_R}$ contains the Majorana neutrinos masses and $\boxed{M_d}$ the remaining quarks and electrons, muon and tau masses, including mixings

I think by now you know the rules. With the algebra and $\boxed{D}$ one builds the one-form, which are the fluctuations of the Dirac operator.The bosonic fields are coming from these one-form $\boxed{\sum_i a_i[D, b_i]}$

Here we run into a problem: the elements of $\boxed{\mathcal{M}_R}$ are the ones which should give rise to the field $\boxed{\sigma}$ as intermediate boson, on a par with the Higgs, and relate to the breaking of the left-right symmetry.

Except that this term either commutes with $\boxed{D}$ or violates the first order condition!

One alternative would is to have a combination of algebra and Dirac operator violating the first order condition

Or we may look for a bigger algebra...

Consider the case of $\boxed{\mathbb{M}_a(\mathbb{H}) \oplus \mathbb{M}_{2a}(\mathbb{C})}$ for the case $\boxed{a = 4}$

In this case we need a $\boxed{2 \cdot (2 \cdot 4)^2 = 128}$ dimensional space, which for 3 generations gives a $\boxed{384}$ dimensional Hilbert space.

I need a representation of the algebra $\boxed{\mathbb{M}_4(\mathbb{H}) \oplus \mathbb{M}_8(\mathbb{C})}$ acting on the spinors I gave earlier, and which can satisfy the stringent order zero conditions

Consider $\boxed{Q \in \mathbb{M}_4(\mathbb{H})}$ and $\boxed{M \in \mathbb{M}_8(\mathbb{C})}$ with indices

$$\boxed{Q^{\dot{t}\beta}_{\dot{s}\alpha} = \begin{pmatrix} Q^{\dot{0}\beta}_{\dot{0}\alpha} & Q^{\dot{1}\beta}_{\dot{0}\alpha} \\ Q^{\dot{0}\beta}_{\dot{1}\alpha} & Q^{\dot{1}\beta}_{\dot{1}\alpha} \end{pmatrix}_{\dot{s}\dot{t}} , \qquad M^{t\mathrm{I}}_{s\mathrm{J}} = \begin{pmatrix} M^{r\mathrm{I}}_{r\mathrm{J}} & M^{l\mathrm{I}}_{r\mathrm{J}} \\ M^{r\mathrm{I}}_{l\mathrm{J}} & M^{l\mathrm{I}}_{l\mathrm{J}} \end{pmatrix}_{st}}$$

where, for any $\boxed{\dot{s}, \dot{t} \in \{\dot{0}, \dot{1}\}}$ and $\boxed{s, t \in \{l, r\}}$, the matrices $\boxed{Q^{\dot{t}\beta}_{\dot{s}\alpha} \in \mathbb{M}_2(\mathbb{H}), \quad M^{t\mathrm{I}}_{s\mathrm{J}} \in \mathbb{M}_4(\mathbb{C})}$ have the index structure defined above

The representation of the element $\boxed{A = (Q, M) \in \mathcal{A}_G}$ is:

$$\boxed{\mathsf{A}^{t\dot{t}\mathsf{C}\;\mathrm{I}\beta}_{s\dot{s}\mathsf{DJ}\alpha} = \left(\delta^{\mathsf{C}}_{\mathsf{0}}\delta^{t}_{s}\delta^{\mathrm{I}}_{\mathsf{J}}\,Q^{\dot{t}\beta}_{\dot{s}\alpha} + \delta^{\mathsf{C}}_{\mathsf{1}}M^{t\mathrm{I}}_{s\mathsf{J}}\delta^{\dot{t}}_{\dot{s}}\delta^{\beta}_{\alpha}\right)}$$

compare with the previous case

$$\boxed{A^{t\dot{t}\,\mathsf{C}\mathrm{I}\beta}_{s\dot{s}\mathsf{DJ}\alpha} = \delta^{t}_{s}\delta^{\dot{t}}_{\dot{s}}\left(\delta^{\mathsf{C}}_{\mathsf{0}}\delta^{\mathrm{I}}_{\mathsf{J}}Q^{\beta}_{\alpha} + \delta^{\mathsf{C}}_{\mathsf{1}}M^{\mathrm{I}}_{\mathsf{J}}\delta^{\beta}_{\alpha}\right)}$$

The spinor indices and the internal gauge indices are mixed. We are in a phase in which the Euclidean structure of space time has not yet emerged.

The fermions are not yet fermions

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