Planck's Inconstant

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It is by now a common belief that at the Panck's scale (or even before) *something is happening...*

We certainly do not what is actually happening, but probably we all agree that the fundamental structure of spacetime will be altered.

In the last years there have been several attempts involving non-trivial commutation relations for the position operator:

$$\boxed{[x^i, x^j] = i\theta^{ij}}$$

or the ones leading to the generalized uncertainty principle

$$\boxed{[x^i, p_j] = i\hbar\left(1 + F^i_j\right)}$$

In general we can consider a more generic space adding also nontrivial commutation relations for momenta

$$\boxed{[p_i, p_j] = iC_{ij}}$$

when $\boxed{\theta, F, C}$ are constant it is easy to express the noncommutativity via a nontrivial $\boxed{\star}$ product, such as the Moyal product or its generalizations

In case they are not constant the explicit construction of the product is much more difficult, although formally possible

The problem is in the symmetries

Even in the simple case of a constant $\boxed{\theta}$ there is no way to preserve rotation (or Lorentz) invariance

The presence of a tensor defines always (except for the trivial case of two dimensions) some preferred directions, which break the symmetry

Less trivial constructions are based on invariance under a quantum group (Hopf algebra)

In more sophisticated approaches (DFR) the operators do not break Lorentz invariance, but preferred directions are picked from the vacuum

Other visions of quantum spacetime point to a granularity of it, such a Loop Quantum Gravity, or changing dimensions, or dual field theories and so on

In all cases we should ask ourselves, operatively the following question:

How is space(time) measured?

In the following I will discuss the question in the context of non relativistic quantum mechanics. A treatment using quantum field theory and/or relativity is more ambitious, but not unreachable

As starting points I will have the observables. They are the selfadjoint part of an algebra of operators which I represent on an Hilbert space.

A state of a physical system is a map from the algebra which is positive and of unit norm. Pure states (which cannot be written as convex sum of other states) are the vectios of the Hilbert space, the rest of the states are represented by mixed density matrices

Usually as algebra we take the (bounded) operators functions of $\boxed{\widehat{x}}$ and $\boxed{\widehat{p}}$

In this view configuration space emerges as the selfadjoint part of a commutative subalgebra, in other word the algebra generated by $\boxed{\widehat{x}}$ alone. From this commutative algebra it is possible to reconstruct the topology of configuration space as the set of pure states of this commutative algebra

Let me go back to the nontrivial commutation relations:

$$[x^i, p_j] = iH^i_j \qquad [x^i, x^j] = i\theta^{ij} \qquad [p_i, p_j] = iC_{ij}$$

The last two relations break rotation invariance

Usually $H^i_j = \hbar\delta^i_j$, otherwise we rescale the coordinates.

Side remark: The natural scale for the length $\sqrt{\theta}$ and the momentum $\sqrt{C}$ is Planckian, which is microscopic for length $\sqrt{\frac{\hbar G}{c^3}} \simeq 10^{-35}\text{m}$, less so for momentum $\sqrt{\frac{\hbar c^3}{G}} \simeq 6.5$ Kg m/sec

One way to recover, at a certain level, the lost invariance is to have $\boxed{\theta}$ depend on space and/or time and to be a *random variables*, fast oscillating around zero

In this case the invariance is recovered in an effective way, as an average

On the other side it is most natural to immagine che the correlation lenght and time of a variable concerning quatum space time be of Planckian nature

Let me a final ingredient to the discussion.

From what I said earlier, in quantum mechanics, what counts is an algebra of observables, and the fact that within it one can recognize a commutative subalgebra defining configuration space

Considering all of the variables of space time as a single vector, $\boxed{Y^A = \{x^i, p_j\}}$, $\boxed{A = 1 \dots 2d}$ the non trivial commutations I described earlier can be stored into a single antysymmetric $\boxed{2d \times 2d}$ matrix

$$\boxed{\Omega = \begin{pmatrix} \theta^{ij} & H^i_j \\ -H^i_j & C_{ij} \end{pmatrix}}$$

It is always possible, at lead locally, to put the matrix $\boxed{\Omega}$ in canonical form with a Darboux transformation to obtain

$$\boxed{\Omega' = \begin{pmatrix} 0 & {H'}^i_j \\ -{H'}^i_j & 0 \end{pmatrix}}$$

This suggest to consider the modified commutation relation

$$\boxed{[x^i, p_j] = {H'}^i_j}$$

Which is tantamount to having a *Planck' Inconstant*

Having fundamental constants of nature to be variables is not a new idea, it goes back to Dirac

But usually the variability was over long (cosmological) time scales. The previous reasoning instead suggests an effective Panck constant with a variability over very shot scales.

In the following we will concentrate on a simple one dimensional model

$$[\mathbf{x}, \mathbf{p}] = i\hbar(1 + \varepsilon(t))$$

with ε rapidly changing with time

$$\overline{\varepsilon(t)} = 0 \quad ; \quad \overline{\varepsilon(t)\varepsilon(t')} = \tau\, \delta(t - t')$$

Overline denotes the mean over ε probability distribution. The fluctuations are uncorrelated for time differences larger than a typical correlation time τ

The time evolution of an operator is

$$\boxed{\frac{d\mathbf{A}}{dt} = \frac{1}{i\hbar}[\mathbf{A}, \mathbf{H}] + \frac{\partial \mathbf{A}}{\partial t}}$$

Here we are assuming that the dependence on $\boxed{\varepsilon}$ is given by the commutator. This means that also the Poisson bracket, whose quantization gives the commutator, is also fluctuating.

This is coherent with the view that the fluctuations are an effective way to take into account an underlying structure

We now need to represent x and p as operators reflecting the modified commutator

$$\mathbf{x}\,\psi(x) = A(t)\,x\,\psi(x) = A(t)\,\mathbf{x}_0\,\psi(x)$$

$$\mathbf{p}\,\psi(x) = -i\hbar\,B(t)\,\frac{d}{dx}\,\psi(x) = B(t)\,\mathbf{p}_0\,\psi(x)$$

with $A(t)B(t) = 1 + \varepsilon(t)$, and $\mathbf{x}_0,\ \mathbf{p}_0$ the canonical pair of standard quantum mechanics.

We treat position and momentum in the same way and choose

$$A(t) = B(t) = \sqrt{1 + \varepsilon(t)}\,,$$

Such an effective variable $\boxed{\hbar}$ will undoubtedly have consequences at several levels. The effects will depend on the scale $\boxed{\tau}$

In our paper we investigated two possible experimental signatures. Surely there will be many more, and we hope that other groups will explore other possibilities. We looked at

- Free particles and interferometric experiments.

- Harmonic Oscillators and coherent light

The Schrödinger equation for the free particle will show a time dependence via $\boxed{\varepsilon(t)}$

$$\boxed{i\hbar\frac{\partial}{\partial t}\psi = \frac{1}{2m}(1+\varepsilon(t))\,\mathbf{p}_0^2\,\psi}$$

It is possible to solve it for a plane wave of momentum $\boxed{p_0}$

$$\boxed{\psi_{p_0}(x,t) = \frac{1}{\sqrt{2\pi}}\exp\left[i\frac{p_0 x}{\hbar} - i\frac{p_0^2}{2m\hbar}\left(t + \int_0^t \varepsilon(t')dt'\right)\right]}$$

When measuring an observable in this scheme there are two averaging processes, conceptually distinct.

- Averaging over the time fluctuations of $\boxed{\varepsilon}$: $\boxed{\bar{\mathbf{A}}}$

- The quantum mechanical averaging. The possible results of a measurement are the eigenvalues of the operator with a probability given by the state: $\boxed{\langle \mathbf{A} \rangle}$

In practice, for $\boxed{\tau}$ smaller than the experimental time resolution the two averaging coincide. Repeating the experiment samples both distribution. Nevertheless they are conceptually different, and I will keep the two notations

For a gaussian peaked at $\boxed{\overline{p}}$ and variance $\boxed{\delta^2}$ we have

$$\boxed{\psi(x,t) = \int \frac{dp_0}{\sqrt{2\pi}} \frac{1}{(\pi\delta^2)^{1/4}} e^{-\frac{(p_0-\overline{p})^2}{2\delta^2} + ip_0 x/\hbar - ip_0^2\left(t + \int_0^t \varepsilon(t)\right)/(2m\hbar)}}$$

The mean distance travelled by a particle is the usual $\boxed{\overline{\langle \mathbf{x} \rangle}_\psi(t) - \overline{\langle \mathbf{x} \rangle}_\psi(0) = \frac{\overline{p}}{m} t\}}$

While the uncertainty is

$$\boxed{(\Delta \mathbf{x})^2_\psi(t) - (\Delta \mathbf{x})^2_\psi(0) = \frac{\delta^2}{2m^2} t^2 + \frac{\overline{p}^2 + \delta^2/2}{m^2} \tau \, t}$$

The motion is like a Brownian motion with diffusion coefficient

$$D = \frac{\overline{p}^2 + \delta^2/2}{2m^2}\tau$$

For $\delta \ll \overline{p}$, one can view D as due to *scatterings* with mean free path $(\overline{p}/m)\tau$.

Scattering over the quantum structure of spacetime

The usual spreading of the wave packet will dominate, but the effect can be enhanced for massless particles. In this case

$$(\Delta \mathrm{x})^2_\psi(t) - (\Delta \mathrm{x})^2_\psi(0) = c^2\, \tau\, t$$

It is possible to measure this effect in a double slit experiment

Waves are detected at some fixed distance L from the plate, the effect is a change δt of travel time with variance

$$\overline{\delta t^2} = \tau\, t = \tau\, L/c$$

with $t = L/c$ the time mean value

For frequency ω and intensity I at the mid-point on the screen

$$I \propto \frac{1}{4}\left| e^{-i\omega(t+\delta t_1)} + e^{-i\omega(t+\delta t_2)} \right|^2 = \frac{1}{2}\left(1 + \cos\left[\omega(\delta t_1 - \delta t_2)\right]\right)$$

$\delta t_{1,2}$ are the uncorrelated time shift along the two paths.In the standard case the two waves show a constructive interference. Here, averaging over $\delta t_{1,2}$

$$I \propto \frac{1}{2}\left(1 + e^{-\omega^2 \tau L/c}\right)$$

For large $\boxed{L, t}$ the intensity behaves as the two waves *were not interfering*

The relevant parameter here is $\boxed{\omega^2 \tau L/c \geq 1}$.

A preliminary analysis puts for Virgo, whose sensibility is bound by the shot noise, a bound

$$\boxed{\tau < 10^{-10} \mathrm{GeV}^{-1} \hbar}$$

For Harmonic Oscillator,we have a variable mass $\boxed{M = m/(1+\varepsilon)}$ and frequency $\boxed{\Omega = \omega(1+\varepsilon)}$ with $\boxed{M\Omega = m\omega}$ constant.

$$\boxed{\mathbf{H} = \frac{1}{2m}(1+\varepsilon)\mathbf{p}_0^2 + \frac{m\omega^2}{2}(1+\varepsilon)\mathbf{x}_0^2 = \frac{1}{2M}\mathbf{p}_0^2 + \frac{M\Omega^2}{2}\mathbf{x}_0^2}$$

The Hamiltonian depends on time via an overall multiplicative factor, and $\boxed{[\mathbf{H}(t), \mathbf{H}(t')] = 0}$, so that Dyson series for time evolution operator can be integrated

$$\boxed{U(t) = \exp\left(-\frac{i}{\hbar}\int_0^t H(t')dt'\right)}$$

Defining, with standard normalization, creation operator

$$\boxed{\mathbf{a} = \sqrt{\frac{m\omega}{2\hbar}}\mathbf{x}_0 + i\frac{1}{\sqrt{2m\hbar\omega}}\mathbf{p}_0}$$

we have $\frac{d\mathbf{a}(t)}{dt} = -i\omega\,(1+\epsilon(t))\,\mathbf{a}(t)$

which has the formal solution

$$\mathbf{a}(t) = \mathbf{a}(0)e^{-i\,\omega t}\sum_n \frac{(-i\omega)^n}{n!}\int_0^t dt_1 \ldots . \int_0^t dt_n \epsilon(t_1) \ldots . \epsilon(t_n)$$

Averaging over $\epsilon(t)$ probability distribution and computing n-point correlation functions in terms of two–point correlation

$$\overline{\mathbf{a}(t)} = \mathbf{a}(0)e^{-i\,\omega t}\sum_k \frac{(-\omega^2\tau t)^k}{2^k k!} = \mathbf{a}(0)e^{-i\,\omega t}e^{-\omega^2\tau t/2}$$

Apart from standard oscillatory term, evolution is exponentially damped on time–scales larger than the characteristic time $2(\omega^2\tau)^{-1}$

Consider a coherent state $\boxed{|\lambda\rangle}$ at $\boxed{t=0}$. Take it real

The evolution of the operators and uncertainty can be calculated

$$\boxed{\overline{\langle \mathbf{x} \rangle}_\lambda(t) = \sqrt{\frac{2\hbar}{m\omega}}\,\lambda\,\cos(\omega t)e^{-\omega^2\tau t/2}} \quad \boxed{\overline{\langle \mathbf{p} \rangle}_\lambda(t) = -\sqrt{2\hbar m\omega}\,\lambda\,\sin(\omega t)e^{-\omega^2\tau t/2}}$$

$$\boxed{\overline{\langle \mathbf{x^2} \rangle}_\lambda(t) = \frac{\hbar}{m\omega}\left[\frac{1}{2} + \lambda^2\left(1 + \cos(2\omega t)e^{-\omega^2\tau t}\right)\right]}$$

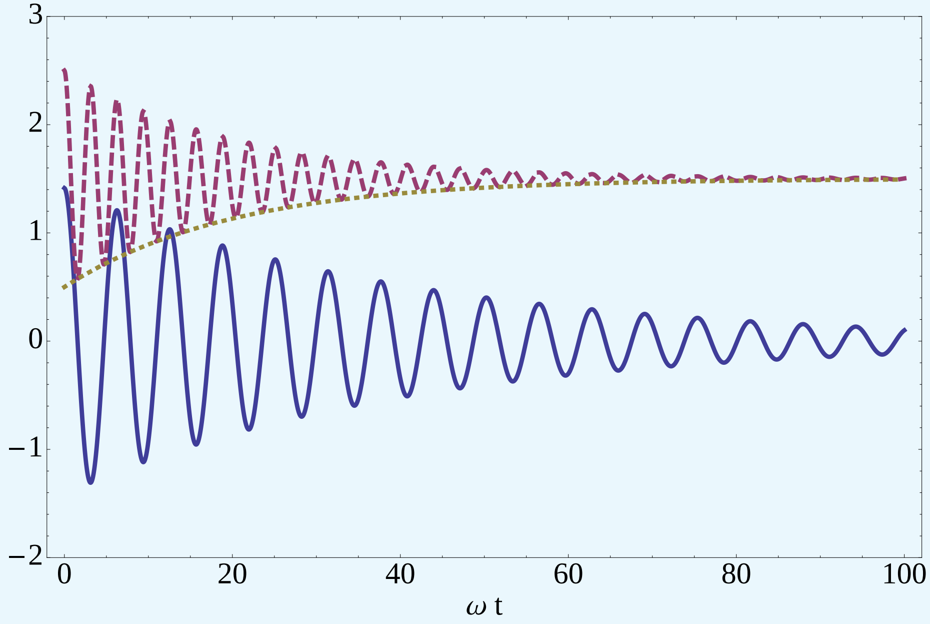
$$\boxed{\overline{\langle \mathbf{p^2} \rangle}_\lambda(t) = m\hbar\omega\left[\frac{1}{2} + \lambda^2\left(1 - \cos(2\omega t)e^{-\omega^2\tau t}\right)\right]}$$

The lower limit of Heisenberg relation is not saturated for $\boxed{\lambda \neq 0}$

$$\boxed{\overline{\Delta\mathbf{x}}_\lambda\overline{\Delta\mathbf{p}}_\lambda = \frac{\hbar}{2}\left[1 + 2\lambda^2\left(1 - e^{-\omega^2\tau t}\right)\right]}$$

This is similar to decoherence processes

The time evolution of position mean value $\overline{\langle \mathbf{x} \rangle}_\lambda$ (solid), $\overline{\langle \mathbf{x^2} \rangle}_\lambda$ (long-dashed) and squared uncertainty $\overline{\Delta \mathbf{x}^2}_\lambda$ (short-dashed) for $\lambda = 1$ in units of appropriate powers of $\sqrt{\hbar/(m\omega)}$.



We choose an unrealistic large value $\omega\tau = 0.05$

This behavior can be translated in the capability of an optical coherent state to survive in a cavity

The effect of the variable $\boxed{\hbar}$ is that the state decoherentizes.

This must be compared with the fact that in any case coherent states in real cavities do not last forever. The two competing effects are however different for scales and functional dependence on time

Present technology, without dedicated experiments, give an order of magnitude for the bound to be

$$\boxed{\tau < 10^{-8}\text{GeV}^{-1}\hbar}$$

Discussion

There some elements, premises of this work, which can have developments

- Lorentz invariance may be the result of granular random structure on very short distances. This is not new (random lattices, causal sets), but it has not been used in NCG

- A quantum space time can give as effective theory, at low energy, one for which some constants of nature are actually variables

- In addition to particle, cosmological and astrophysical experiments, there could be signatures of quantum spacetime (non necessarily noncommutative) that can be seen using new kinds of experiments, such as interferometry, or optical cavities