Spectral Action, a status report

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I will make some considerations about relevant physical aspects of the general framework of spectral geometry

The framework in which I will present the work is that of the spectral triples, i.e. the approach to geometry based on the spectral properties of the algebra of operators defined upon them. The construction is the one mathematically needed to generalize ordinary geometry to noncommutative geometry

In particular I will have in mind the phenomenological consequences of this approach in the “LHC era”.

The starting point of Connes' approach to is that geometry and its (noncommutative) generalizations are described by the spectral data of three basic ingredients:

- An algebra $\boxed{\mathcal{A}}$ which describes the topology of spacetime.
- An Hilbert space $\boxed{\mathcal{H}}$ on which the algebra act as operators, and which also describes the matter fields of the theory.
- A (generalized) Dirac Operator $\boxed{D_0}$ which carries all the information of the metric structure of the space, as well as other crucial information about the fermions.

An important role is also played by two other operators: the chirality $\boxed{\gamma}$ and charge conjugation $\boxed{J}$

There is a theorem (Gefand-Najmark): the category of commmutative $\boxed{C^*}$-algebras and that of topological Hausdorff spaces are in one to one correspondence. The algebra being that of continuous complex valued functions on the space.

Connes programme is the transcription of all usual geometrical objects into algebraic terms, so to provide a ready generalization to the case for which the algebra is noncommutative

The points of the space (that can be reconstruced) are pure states, or maximal ideals of the algebra, or irreducible representations. They all coincide in the commutative case.

The metric aspects are encoded in the Dirac operator. One forms are represented by operators of the kind $\boxed{a[D_0, b]}$. Bundles are projective module . . .

The construction the dictionary is progressing encompassing most of geometry. And making it ready for the noncommutative generalization

While the formalism is geared towards the construction of genuine noncommutative spaces, spectacular results are obtained considering almost commutative geometries, which leads to: **Connes' approach to the standard model**

The project is to transcribe electrodynamics on an ordinary manifold using algebraic concepts: The algebra of functions, the Dirac operator, the Hilbert space and chirality and charge conjugation. One can then write the action in purely algebraic terms.

Then the machinery can be applied to noncommutative space, or in general to other algebras.

Remarkably, if one applies this to the algebra of functions valued in diagonal $\boxed{2 \times 2}$ matrices one finds the Higgs Lagrangian of a $\boxed{U(1) \times U(1) \to U(1)}$ breaking, in which the Higgs is the "vector" boson corresponding to the internal degree of freedom.

In this case the space is only "almost" noncommutative, in the sense that there still is an underlying spacetime, the noncommutative algebra describing space is said to be Morita equivalent to a commutative algebra

For the full standard the algebra is a tensor product $\boxed{\mathcal{A} = C(\mathbb{R}^4) \otimes \mathcal{A}_F}$, with $\boxed{\mathcal{A}_F}$ a finite matrix algebra of $\boxed{3 \times 3}$ matrices, quaternions (which are matrices of the kind $\boxed{a^\mu \sigma_\mu}$) and complex numbers corresponding to $\boxed{SU(3), SU(2)}$ and $\boxed{U(1)}$ respectively.

The information about masses and Cabibbo mixing are encoded in the $\boxed{D}$ operator

The algebraic concepts are more robust than those based on "pointwise" geometry and they survive when the algebra is noncommutative.

In the commutative case it is possible to characterize a manifold with properties of the elements of the triple (all five of them)

There is a list of conditions and a theorem (Connes) which proves this.

Since the conditions are all purely algebraic they remain valid in the noncommutative case, defining a noncommutative manifold

1. Dimension There is a nonnegative integer n such that the operator the Diximier trace of $|D_0|^{-1n}$ is finite.

2. Regularity For any $a \in \mathcal{A}$ both a and $[D_0, a]$ belong to the domain of δ^k for any integer k, where δ is the derivation given by $\delta(T) = [|D|, T]$.

3. Finiteness The space $\bigcap_k \mathrm{Dom}(D^k)$ is a finitely generated projective left $\mathcal{A}$ module.

4. Reality There exist J with the commutation relation fixed by the number of dimensions with the property

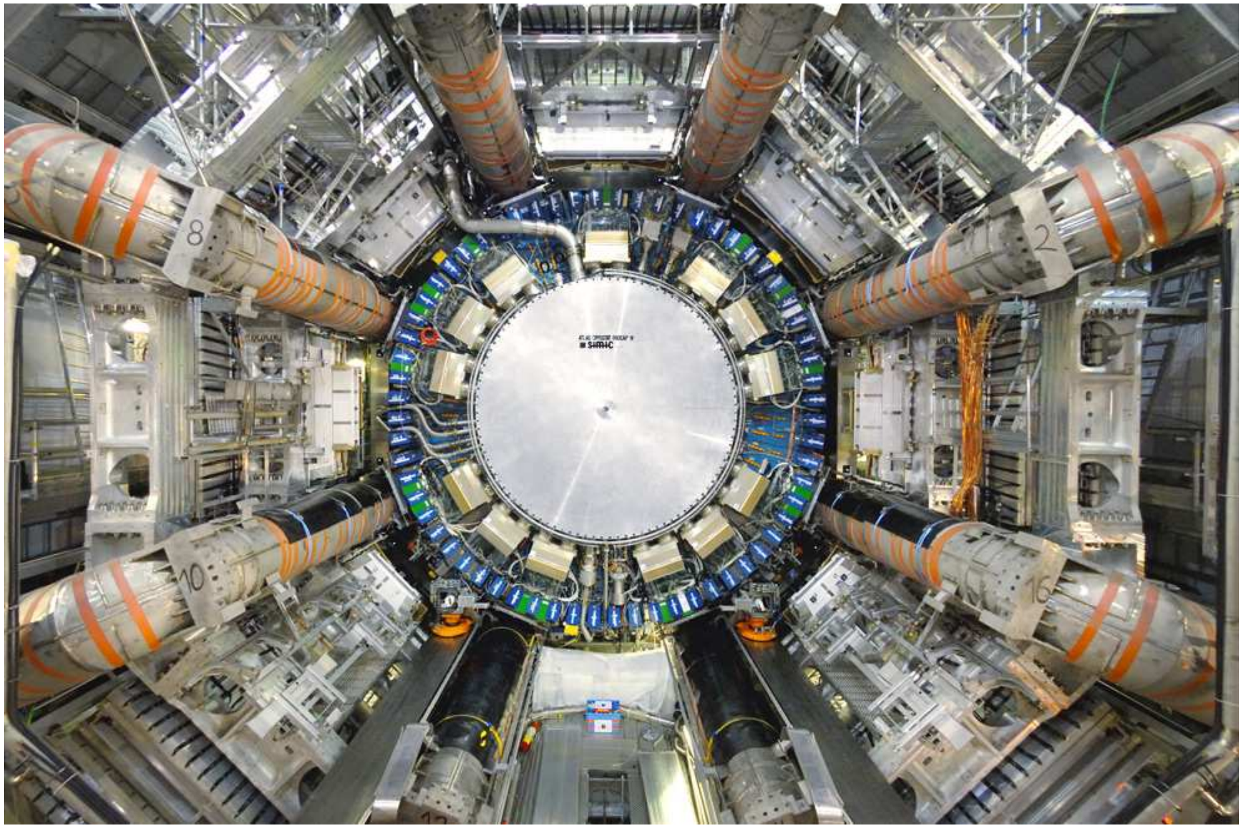
   (a) Commutant $[a, Jb^* J^{-1}] = 0, \forall a, b$

   (b) First order $[[D, a], b^o = Jb^* J^{-1}] = 0\ , \forall a, b$

5. Orientation There exists a Hochschild cycle c of degree n which gives the grading γ , This condition gives an abstract volume form.

6. Poincaré duality A Certain intersection form detemrined by D_0 and by the K-theory of $\mathcal{A}$ and its opposite is nondegenrate.

So what has this to do with the **Large Hadron Collider** ?



A particularly simple form of noncommutative geometry describes the standard model of particle interaction.

The noncommutative geometry is particularly simple because it is the product of an infinite dimensional commutative algebra times a noncommutative finite dimensional one.

This describes a mild generalization of the space, and it is natural to think that the standard model is an effective theory.

The infinite dimensional part is the one relative to the four dimensional space-time

I have to mention that everything works only if this spacetime is compact and Euclidean. At present there are not complete and satisfactory Minkowskian version of nocommutative geometry.

We start from the algebra, a tensor product $\boxed{\mathcal{A} = C(\mathbb{R}^4) \otimes \mathcal{A}_F}$

As Hilbert space we take the tensor product of spinor times the degree of freedom of the known fermions, including right-handed neutrinos: 96 degrees of freedom (16 fermions for 3 families for 2 spin degree).

$$\boxed{\mathcal{H} = sp(L^2(\mathbb{R}^4)) \otimes \mathcal{H}_F}$$

The chiral and real structure are $\boxed{\Gamma = \gamma^5 \otimes \gamma_F}$ and $\boxed{\mathcal{J} = J \otimes J_F}$ where the finite part are finite matrices. $\boxed{\gamma_F}$ is $\boxed{\pm 1}$ on particles/antiparticle, and J_F excahnges them and acts as complex conjugation

The Dirac operator is

$$\boxed{D = \partial\!\!\!/ \otimes \mathbb{I} + \gamma^5 \otimes D_F}$$

$$\boxed{D_F = \begin{pmatrix} \partial\!\!\!/ & \gamma^5 \mathcal{M} & \gamma^5 \mathcal{M}_R & 0 \\ \gamma^5 \mathcal{M}^\dagger & \partial\!\!\!/ & 0 & 0 \\ \gamma^5 \mathcal{M}_R^\dagger & 0 & \partial\!\!\!/ & \gamma^5 \mathcal{M}^* \\ 0 & 0 & \gamma^5 \mathcal{M}^T & \partial\!\!\!/ \end{pmatrix}}$$

$\boxed{\mathcal{M}}$ contains the Dirac masses, or rather, the Yukawa couplings,
$\boxed{\mathcal{M}_R}$ contains Majorana masses.

We need to impose on the algebra $\boxed{\mathcal{A}_F}$ that it is a noncommutative manifold. An application of the seven conditions singles out the finite dimensional algebras that satisfy the requisites:

$$\boxed{M_a(\mathbb{H}) \oplus M_{2a}(\mathbb{C})}$$

acting on a $\boxed{2(2a)^2}$ dimensional Hilbert space

Notice that since the Hilbert space is the tensor product of four dimensional spinors by the 96-dimensional elements of $\mathcal{H}_F$, the elements of the Hilbert space have in reality 384 dimension, 128 for a single generation. This redundancy of states is known as fermion doubling.

This Dirac operator reproduces the classical fermionic action with fermion masses. The bosonic part of the action, and more importantly the quantization of the fields can be achieved via the spectral action principle.

The simplest nontrivial possibility is $\boxed{\mathcal{A}_F = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})}$

The other conditions, and especially the presence of a neutrino Majorana mass reduce the algebra to the standard model algebra

$$\boxed{\mathcal{A}_{sm} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})}$$

The unitaries of the algebra correspond to the symmetries of the standard model: $\boxed{SU(3) \oplus SU(2) \oplus U(1)}$

A unimodularity condition takes care of the extra U(1)

This algebra must be represented as operators on a Hilbert space, which also has a continuos infinite dimensional part (spinors on spacetime) times a finite dimensional one: $\boxed{\mathcal{H} = \mathrm{sp}(\mathbb{R}) \otimes \mathcal{H}_F}$. The grading given by $\boxed{\gamma}$ splits it into a left and right subspace: $\boxed{\mathcal{H}_L \oplus \mathcal{H}_R}$

The $\boxed{J}$ operator basically exchange the two chiralities and conjugates, thus effectively making the algebra act form the right.

For $\boxed{\mathcal{H}_{\mathcal{F}}}$ we take known fermions: In total there are 96 degrees of freedom, times the 4 dimensions of the spinors this makes 384.

Take this complicated Hilbert space, and find a representation of $\boxed{\mathcal{A}_F}$ in such a way that the fermions transform properly and the conditions involving $\boxed{\mathcal{A}, \gamma}$ and $\boxed{J}$ are satisfied

Then one has to define a significant $\boxed{D_0}$ which will satisfy the remaining conditions, and which will have physical relevance (more later).

The good news is that this is possible. The standard model is possible. Other unified models do not satisfy the stringent requirements.

The $\boxed{D_0}$ operator carries the metric information on the continuous part as well as the internal part.

$$\boxed{D_0 = \gamma^\mu(\partial_\mu + \omega_\mu) \otimes \mathbb{I} + \gamma^5 \otimes D_F}$$

$\boxed{\omega_\mu}$ the spin connection. The presence of $\boxed{\gamma^5}$, the chirality operator for the continuous manifold is for the Euclidean signature.

All of the properties of the internal part are encoded in $\boxed{D_F}$, which is a $\boxed{96 \times 96}$ matrix.

With $\boxed{D_0}$ one then builds the generic connection one-forms $\boxed{A = \sum_i a_i[D_0, b_i]}$, and the fluctuations $\boxed{D_A = D_0 + A + JAJ}$

In the spirit of noncommutative geometry the action must be spectral. It comprise of a fermionic and a bosonic part

$$\boxed{S_F = \langle \Psi | \, D_A \, | \Psi \rangle}$$

$$\boxed{S_B = \text{Tr}\, \chi \left(\frac{D_A}{\Lambda} \right)}$$

where $\boxed{D_A = D_0 + A}$ is a fluctuation of the Dirac operator, $\boxed{\chi}$ is the characteristic function of the interval $\boxed{[0,1]}$, or some smoothened version of it, and $\boxed{\Lambda}$ is a cutoff

Then there is a "standard" fermionic action $\boxed{\langle \Psi | \, D_A \, | \Psi \rangle}$

The bosonic action is finite by construction, the fermionic part needs to be regularized

Technically the bosonic spectral action is a sum of residues and can be expanded in a power series in terms of Λ^{-1} as

$$S_B = \sum_n f_n \, a_n(D^2/\Lambda^2)$$

where the f_n are the momenta of χ

$$\begin{aligned} f_0 &= \int_0^\infty \mathrm{d}x \, x\chi(x) \\ f_2 &= \int_0^\infty \mathrm{d}x \, \chi(x) \\ f_{2n+4} &= (-1)^n \partial_x^n \chi(x)\Big|_{x=0} \quad n \geq 0 \end{aligned}$$

the a_n are the Seeley-de Witt coefficients which vanish for n odd. For D^2 of the form

$$D^2 = -(g^{\mu\nu}\partial_\mu\partial_\nu\mathbb{I} + \alpha^\mu\partial_\mu + \beta)$$

defining (in term of a generalized spin connection containing also the gauge fields)

$$\begin{aligned}
\omega_\mu &= \frac{1}{2} g_{\mu\nu} \left(\alpha^\nu + g^{\sigma\rho}\Gamma^\nu_{\sigma\rho}\mathbb{I}\right) \\
\Omega_{\mu\nu} &= \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu] \\
E &= \beta - g^{\mu\nu}\left(\partial_\mu\omega_\nu + \omega_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho\right)
\end{aligned}$$

then

$$\begin{aligned}
a_0 &= \frac{\Lambda^4}{16\pi^2}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,\mathbb{I}_F \\
a_2 &= \frac{\Lambda^2}{16\pi^2}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\left(-\frac{R}{6} + E\right) \\
a_4 &= \frac{1}{16\pi^2}\frac{1}{360}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,(-12\nabla^\mu\nabla_\mu R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} \\
&\quad +2R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 60RE + 180E^2 + 60\nabla^\mu\nabla_\mu E + 30\Omega_{\mu\nu}\Omega^{\mu\nu})
\end{aligned}$$

tr is the trace over the inner indices of the finite algebra $\mathcal{A}_F$ and in Ω and E are contained the gauge degrees of freedom including the gauge stress energy tensors and the Higgs, which is given by the inner fluctuations of D

We now apply this machinery to the standard model to find the Lagrangian of the standard model coupled to gravitation

It is a straightforward calculation, one has to just crank a machine

The cranking depends of course on the machine.

The final result is:

$$
\begin{gathered}
\mathcal{L}_{SM} = -\frac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \\
\frac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \frac{1}{2c_w^2} M^2 Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - igc_w(\partial_\nu Z^0_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu (W^+_\mu \partial_\nu W^-_\mu - \\
W^-_\mu \partial_\nu W^+_\mu) + Z^0_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu (W^+_\mu \partial_\nu W^-_\mu - \\
W^-_\mu \partial_\nu W^+_\mu) + A_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - \frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + \\
g^2 c_w^2 (Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2 (A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w (A_\mu Z^0_\nu (W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \frac{1}{2}\partial_\mu H \partial_\mu H - 2M^2 \alpha_h H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \\
\beta_h \left(\frac{2M^2}{g^2} + \frac{2M}{g} H + \frac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)\right) + \frac{2M^4}{g^2}\alpha_h - g\alpha_h M \left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-\right) - \\
\frac{1}{8}g^2 \alpha_h \left(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2 H^2\right) - gMW^+_\mu W^-_\mu H - \\
\frac{1}{2}g\frac{M}{c_w^2} Z^0_\mu Z^0_\mu H - \frac{1}{2}ig \left(W^+_\mu (\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) - W^-_\mu (\phi^0 \partial_\mu \phi^+ - \phi^+ \partial_\mu \phi^0)\right) + \\
\frac{1}{2}g \left(W^+_\mu (H\partial_\mu \phi^- - \phi^- \partial_\mu H) + W^-_\mu (H\partial_\mu \phi^+ - \phi^+ \partial_\mu H)\right) + \frac{1}{2}g\frac{1}{c_w}(Z^0_\mu (H\partial_\mu \phi^0 - \phi^0 \partial_\mu H) + \\
M\,(\frac{1}{c_w} Z^0_\mu \partial_\mu \phi^0 + W^+_\mu \partial_\mu \phi^- + W^-_\mu \partial_\mu \phi^+) - ig\frac{s_w^2}{c_w} M Z^0_\mu (W^+_\mu \phi^- - W^-_\mu \phi^+) + igs_w M A_\mu (W^+_\mu \phi^- - W^-_\mu \phi^+) - \\
ig\frac{1-2c_w^2}{2c_w} Z^0_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) + igs_w A_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) - \frac{1}{4}g^2 W^+_\mu W^-_\mu \left(H^2 + (\phi^0)^2 + 2\phi^+\phi^-\right) - \\
\frac{1}{8}g^2 \frac{1}{c_w^2} Z^0_\mu Z^0_\mu \left(H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2 \phi^+\phi^-\right) - \frac{1}{2}g^2 \frac{s_w^2}{c_w} Z^0_\mu \phi^0 (W^+_\mu \phi^- + W^-_\mu \phi^+) - \\
\frac{1}{2}ig^2 \frac{s_w^2}{c_w} Z^0_\mu H (W^+_\mu \phi^- - W^-_\mu \phi^+) + \frac{1}{2}g^2 s_w A_\mu \phi^0 (W^+_\mu \phi^- + W^-_\mu \phi^+) + \frac{1}{2}ig^2 s_w A_\mu H (W^+_\mu \phi^- - W^-_\mu \phi^+) - \\
g^2 \frac{s_w}{c_w}(2c_w^2 - 1) Z^0_\mu A_\mu \phi^+\phi^- - g^2 s_w^2 A_\mu A_\mu \phi^+\phi^- + \frac{1}{2}ig_s\, \lambda^a_{ij} (\bar{q}^\sigma_i \gamma^\mu q^\sigma_j) g^a_\mu - \bar{e}^\lambda (\gamma\partial + m^\lambda_e) e^\lambda - \bar{\nu}^\lambda (\gamma\partial + \\
m^\lambda_\nu)\nu^\lambda - \bar{u}^\lambda_j (\gamma\partial + m^\lambda_u) u^\lambda_j - \bar{d}^\lambda_j (\gamma\partial + m^\lambda_d) d^\lambda_j + igs_w A_\mu \left(-(\bar{e}^\lambda \gamma^\mu e^\lambda) + \frac{2}{3}(\bar{u}^\lambda_j \gamma^\mu u^\lambda_j) - \frac{1}{3}(\bar{d}^\lambda_j \gamma^\mu d^\lambda_j)\right) + \\
\frac{ig}{4c_w} Z^0_\mu \{(\bar{\nu}^\lambda \gamma^\mu (1+\gamma^5)\nu^\lambda) + (\bar{e}^\lambda \gamma^\mu (4s_w^2 - 1 - \gamma^5) e^\lambda) + (\bar{d}^\lambda_j \gamma^\mu (\frac{4}{3}s_w^2 - 1 - \gamma^5) d^\lambda_j) + (\bar{u}^\lambda_j \gamma^\mu (1 - \frac{8}{3}s_w^2 + \\
\gamma^5) u^\lambda_j)\} + \frac{ig}{2\sqrt{2}} W^+_\mu \left((\bar{\nu}^\lambda \gamma^\mu (1+\gamma^5) {U^{lep}}_{\lambda\kappa} e^\kappa) + (\bar{u}^\lambda_j \gamma^\mu (1+\gamma^5) C_{\lambda\kappa} d^\kappa_j)\right) + \\
\frac{ig}{2\sqrt{2}} W^-_\mu \left((\bar{e}^\kappa {U^{lep}}^\dagger_{\kappa\lambda} \gamma^\mu (1+\gamma^5) \nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\kappa\lambda} \gamma^\mu (1+\gamma^5) u^\lambda_j)\right) + \\
\frac{ig}{2M\sqrt{2}} \phi^+ \left(-m^\kappa_e (\bar{\nu}^\lambda {U^{lep}}_{\lambda\kappa} (1-\gamma^5) e^\kappa) + m^\lambda_\nu (\bar{\nu}^\lambda {U^{lep}}_{\lambda\kappa} (1+\gamma^5) e^\kappa)\right) + \\
\frac{ig}{2M\sqrt{2}} \phi^- \left(m^\lambda_e (\bar{e}^\lambda {U^{lep}}^\dagger_{\lambda\kappa} (1+\gamma^5) \nu^\kappa) - m^\kappa_\nu (\bar{e}^\lambda {U^{lep}}^\dagger_{\lambda\kappa} (1-\gamma^5) \nu^\kappa)\right) - \frac{g}{2}\frac{m^\lambda_\nu}{M} H (\bar{\nu}^\lambda \nu^\lambda) - \frac{g}{2}\frac{m^\lambda_e}{M} H (\bar{e}^\lambda e^\lambda) + \\
\frac{ig}{2}\frac{m^\lambda_\nu}{M} \phi^0 (\bar{\nu}^\lambda \gamma^5 \nu^\lambda) - \frac{ig}{2}\frac{m^\lambda_e}{M} \phi^0 (\bar{e}^\lambda \gamma^5 e^\lambda) - \frac{1}{4}\, \bar{\nu}_\lambda\, M^R_{\lambda\kappa}\, (1-\gamma_5) \hat{\nu}_\kappa - \frac{1}{4}\, \overline{\bar{\nu}_\lambda\, M^R_{\lambda\kappa}\, (1-\gamma_5) \hat{\nu}_\kappa} + \\
\frac{ig}{2M\sqrt{2}} \phi^+ \left(-m^\kappa_d (\bar{u}^\lambda_j C_{\lambda\kappa} (1-\gamma^5) d^\kappa_j) + m^\lambda_u (\bar{u}^\lambda_j C_{\lambda\kappa} (1+\gamma^5) d^\kappa_j)\right) + \\
\frac{ig}{2M\sqrt{2}} \phi^- \left(m^\lambda_d (\bar{d}^\lambda_j C^\dagger_{\lambda\kappa} (1+\gamma^5) u^\kappa_j) - m^\kappa_u (\bar{d}^\lambda_j C^\dagger_{\lambda\kappa} (1-\gamma^5) u^\kappa_j)\right) - \frac{g}{2}\frac{m^\lambda_u}{M} H (\bar{u}^\lambda_j u^\lambda_j) - \frac{g}{2}\frac{m^\lambda_d}{M} H (\bar{d}^\lambda_j d^\lambda_j) + \\
\frac{ig}{2}\frac{m^\lambda_u}{M} \phi^0 (\bar{u}^\lambda_j \gamma^5 u^\lambda_j) - \frac{ig}{2}\frac{m^\lambda_d}{M} \phi^0 (\bar{d}^\lambda_j \gamma^5 d^\lambda_j)
\end{gathered}
$$

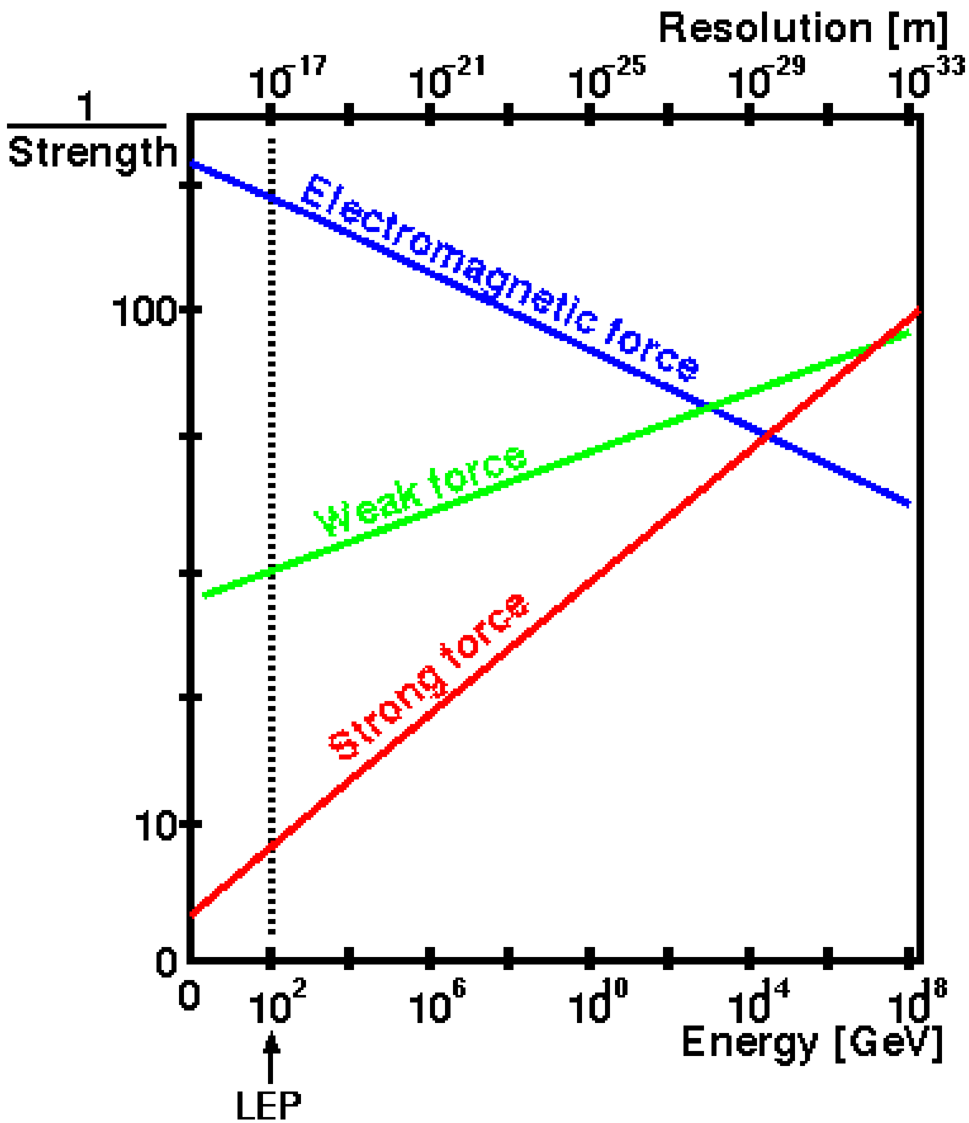
Here the notation is as in [46], as follows.

This is the standard model coupled with gravity in a nonminimal way. I have no time in this talk to discuss all the work that is being done from a cosmological point of view (Sakellariadou, Marcolli, Kurkov . . .

There is one aspect which is important. In the operator $\boxed{D_F}$ all quantities refer to the fermions. There is no mention of the bosons, and in particular of the Higgs

In fact all bosons appear in the connection one form, and this includes the Higgs, which appear as an intermediate boson, on a par with the $\boxed{W, Z}$ and $\boxed{\gamma}$

Another important point is that the model needs all gauge coupling constants to be equal at the scale $\boxed{\Lambda}$, that is taken to be the unification scale. A scale which exists in an approximate sense.


Resolution [m]
10^{-17}
10^{-21}
10^{-25}
10^{-29}
10^{-33}
$\frac{1}{\text{Strength}}$
100
10
0
Electromagnetic force
Weak force
Strong force
0
10^{2}
10^{6}
10^{10}
10^{14}
10^{18}
Energy [GeV]
LEP

The running of the coupling constants of the Higgs, basically the coupling of the quartic term, at one loop is a function of the running of the coupling constants (which is not changed) and the running of the Yukawa couplings of the fermions, which are dominated by the top.

The mass of the Higgs is then, in this model, a fixed combination of the se quantities (one fixes the vev of the Higgs), and therefore is a predicted quantity

The predicted values, which depends on the unification point, is in the vicinity of 170 GeV

As you know this value is not the correct mass of the Higgs, which is 126 GeV. I still find however remarkable the fact that a theory based on profound mathematical requirements, putting togehter quantities over a wide range of scales, comes up with a number which is relatively close.

Can we do better?

C. Stephan has models with a lower mass for the Higgs. However he enlarges the algebra and the Hilbert space. Hence he does not have a manifold, in the noncommutative sense.

Recently Chamseddine and Connes have found a more elegant solution. Recall that in order to have the correct standard model algebra one has to introduce a Majorana mass

The idea is to consider this entry as a new field: $\boxed{\sigma}$

This new field interacts with the other particle in a way similar to the interaction of the field which breaks a left-right chiral symmetry. It acts as a second Higgs and as such alters the running of its parameters

In this case the mass is compatible with 126 GeV

The problem is that in this approach the bosonic fields have a precise origin. They come from the connection one-form.

$$\boxed{A = \sum_i a_i [D_0, b_i]}$$

But if one commutes $\boxed{D_F}$ the entry corresponding to the Majorana mass commutes with elements of the algebra.

The rigidity of the model works against this solution. C&C "promote" this Yukawa coupling to be a field, a procedure not completely justified

A. Devastato, P. Martinetti and I propose a solution: a grand symmetry.

In NCG the usual grand unified groups, such as $\boxed{SU(5)}$ or $\boxed{SO(10)}$ do not work. There are very few representations of algebra as opposed to groups. Finite dimensional algebras only have one nontrivial IRR

Fortunately in the standard model there are only weak doublets and colour triplets, so it works

Recall that a finite "manifold" is an algebra: $\boxed{M_a(\mathbb{H}) \oplus M_{2a}(\mathbb{C})}$ acting on a $\boxed{2(2a)^2}$ dimensional Hilbert space. So far we had $\boxed{a = 2,\ 2(2a)^2 = 32 \times 3}$

For $\boxed{M_4(\mathbb{H}) \oplus M_8(\mathbb{C})}$ one requires a $\boxed{128 \times 3 = 384}$ dimensional space.

This is exactly the dimension of the Hilbert space if we take the fermion doubling into account

It is necessary to look at Hilbert space with different eyes

$$\boxed{\mathcal{H} = sp(L^2(\mathbb{R}^4)) \otimes \mathcal{H}_F = L^2(\mathbb{R}^4) \otimes \mathsf{H}_F}$$

where now the dimensions of $\boxed{\mathsf{H}_F}$ is 384

It is still possible to represent the gran algebra $\boxed{M_4(\mathbb{H}) \oplus M_8(\mathbb{C})}$ satisfying all of the axioms. This is highly nontrivial if one keeps the same Hilbert space.

But this time the algebra does not act diagonally on the spinor indices. it mixes them.

I will not write explicitly the two representation (on particle and anti particles) because they are rather involved, and just state the results

One now considers this algebra to the high energy description, so that the standard model is some sort of effective low energy theory, coming after the breaking due to the Dirac operator

We have two concurrent breakings. We split them in two stages to render simple the exposition

Since the algebra now is not diagonal in the spin indices it is possible to consider a finite Dirac operator which also is not diagonal also, and which has a Majorana mass which gives a non trivial one form.

Together with the order one condition we have the reduction of the grand algebra to

$$\boxed{(\mathbb{C} \oplus \mathbb{H} \oplus \mathbb{M}_3(\mathbb{C})) \oplus \left(\mathbb{C}' \oplus \mathbb{H}' \oplus \mathbb{M}_3'(\mathbb{C})\right)}$$

The one-form which is now present, and is therefore a field, is exactly the $\boxed{\sigma}$ filed which was introduced by hand by Chamseddine and Connes

The different now is that we don not have to promote it to be a filed by hand. It must be a field because is coming form a one-form

It may seem that there are two copies of the standard model. But let me stress that we do not envisage a phase with a doubling of the standard model because this symmetry is broken at the same time by a novel mechanism.

Consider spacetime, as opposed to the internal space. Without the presence of $\boxed{\partial\!\!\!/}$ we have just a topological space. The presence of metric, but also of the Spin structure, i.e. a "Lorentz" structure is in the Dirac operator (actually we are in the Euclidean so we have $\boxed{SO(4)}$.

Since the gran algebra in not anymore diagonal $\boxed{\partial\!\!\!/}$ does not commute with the finite dimensional algebra, so there will be one-form and there may be symmetry breakings

For lack of time let me give just the results:

1) The Dirac operator $\boxed{\partial\!\!\!/}$ causes the breaking of the two copies of the standard model to a single copy.

2) $\boxed{SO(4)}$ emerges as the broken group by some sort of Higgs mechanism, with the spin connection as some sort of Higgs field, i.e. the one connection one-form of the fluctuations around the nontrivial minimum

There are of course still many unpleasant features of this approach: It is Euclidean, it requires unification of the constants at a relatively low scales . . .

On the other side it keeps pace with experiments, and is maturing towards the possibility to be able to say something of phenomenological relevance

ANd it is particularly important for mathematical physics (and the mathematical methods line of research of INFN) since it provides an excellent example of how advanced mathematics has immediate applications to physically relevant frontier problems.