The spectral point of view of interactions, renormalization and the early universe

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I will make some considerations about relevant physical aspects of the general framework of spectral geometry

The framework in which I will present the work is that of the spectral triples, i.e. the approach to geometry based on the spectral properties of the algebra of operators defined upon them. The construction is the one mathematically needed to generalize ordinary geometry to noncommutative geometry

The whole work presented here however could be recast in a way which does not make any explicit mention of noncommutative geometry. I feel nevertheless useful to have in mind the larger picture.

The starting point of Connes' approach to is that geometry and its (noncommutative) generalizations are described by the spectral data of three basic ingredients:

- An algebra $\boxed{\mathcal{A}}$ which describes the topology of spacetime.

- An Hilbert space $\boxed{\mathcal{H}}$ on which the algebra act as operators, and which also describes the matter fields of the theory.

- A (generalized) Dirac Operator $\boxed{D_0}$ which carries all the information of the metric structure of the space, as well as other crucial information about the fermions.

An important role is also played by two other operators: the chirality $\boxed{\gamma}$ and charge conjugation $\boxed{J}$

There is a profound mathematical result (Gefand-Najmark) which states that the category of commmutative $\boxed{C^*}$-algebras and that of topological Hausdorff spaces are in one to one correspondence. The algebra being that of continuous complex valued functions on the space.

Connes programme is the transcription of all usual geometrical objects into algebraic terms, so to provide a ready generalization to the case for which the algebra is noncommutative

The points of the space (that can be reconstruced) are pure states, or maximal ideals of the algebra, or irreducible representations. They all coincide in the commutative case.

The metric aspects are encoded in the Dirac operator. For example the metric distance is given by

$$\boxed{d(x,y) = \sup_{\|[D_0,a]\| \le 1} |a(x) - a(y)|}$$

One forms are represented by operators of the kind $\boxed{a[D_0, b]}$.

Bundles are projective module . . .

The construction the dictionary is progressing encompassing most of geometry. And making ir ready for the noncommutative generalization

While the formalism is geared towards the construction of genuine noncommutative spaces, spectacular results are obtained considering almost commutative geometries, which leads to: **Connes' approach to the standard model**

The project is to transcribe electrodynamics on an ordinary manifold using algebraic concepts: The algebra of functions, the Dirac operator, the Hilbert space and chirality and charge conjugation. One can then write the action in purely algebraic terms.

Then the machinery can be applied to noncommutative space, or in general to other algebras.

Remarkably, if one applies this to the algebra of functions valued in diagonal $\boxed{2 \times 2}$ matrices one finds the Higgs Lagrangian of a $\boxed{U(1) \times U(1) \to U(1)}$ breaking, in which the Higgs is the "vector" boson corresponding to the internal degree of freedom.

In this case the space is only "almost" noncommutative, in the sense that there still is an underlying spacetime, the noncommutative algebra describing space is said to be Morita equivalent to a commutative algebra

For the full standard the algebra is a tensor product $\boxed{\mathcal{A} = C(\mathbf{R}^4) \otimes \mathcal{A}_F}$, with $\boxed{\mathcal{A}_F}$ a finite matrix algebra of $\boxed{3 \times 3}$ matrices, quaternions (which are matrices of the kind $\boxed{a^\mu \sigma_\mu}$) and complex numbers corresponding to $\boxed{SU(3), SU(2)}$ and $\boxed{U(1)}$ respectively.

The information about masses and Cabibbo mixing are encoded in the $\boxed{D}$ operator

There is a translation in algebraic terms of the requirement that a generic topological space is a manifold (i.e. it has a differential structure). This is a set of seven purely algebraic conditions on the algebra, the Hilbert space and the $\boxed{D_0, \gamma}$ and $\boxed{J}$ operators.

Application of these conditions to the almost commutative geometry, plus the imposition of chirality, select the gauge group to be $\boxed{SU(3) \times SU(2) \times U(1)}$.

The model, especially in its last version (Chamseddine-Connes-Marcolli) has predictive power (mass of the Higgs). Many of the actual calculations were made also by the Marseille group, and in the next days Thomas Schucker will give more details.

The presence of chirality $\boxed{\gamma = \gamma^\dagger}$, with $\boxed{\gamma^2 = 1}$, the generalization of $\boxed{\gamma_5}$, causes the splitting

$$\boxed{\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R}$$

Eigenspaces of $\boxed{\frac{1}{2}(1 \pm \gamma)}$

The other operator, $\boxed{J}$, charge conjugation, which however plays no important role in this seminar. it has important mathematical connections, Tomita-Takesaki operator, KMS states etc.

The central idea behind spectral geometry is that these ingredients are sufficient to describe not only a geometry, but also the behaviour of the fields defined on them, and their couplings to the geometry of spacetime (gravity). Treating on an equal footing the *external geometry (spacetime), with the inner one, gauge degrees of freedom*

The main success of this view is the spectral action. The algebra is the product of the algebra of functions on spacetime, the Hilbert space is that of fermion matter fields, and the Dirac operator contains all information on the metric of spacetime, as well as the masses , couplings and mixings of fermions.

The spectral action contains two part, one is the bosonic action, to be read in a Wilsonian renormalization group sense:

$$\boxed{S_B = \mathrm{Tr}\,\chi\left(\frac{D_A}{\Lambda}\right)}$$

where $\boxed{D_A = D_0 + A}$ is a fluctuation of the Dirac operator, $\boxed{\chi}$ is the characteristic function of the interval $\boxed{[0,1]}$, or some smoothened version of it, and $\boxed{\Lambda}$ is a cutoff

Then there is a "standard" fermionic action $\boxed{\langle\Psi|\, D_A \,|\Psi\rangle}$

The bosonic action is finite by construction, the fermionic part needs to be regularized

In the work of Chamseddine, Connes and Marcolli the renormalization group flow is done by considering as boundary condition the unification of e three interaction coupling constants at $\boxed{\Lambda}$. This is approximately (but not exactly) true.

The various couplings and parameters are then found at low energy via the renormalization flow

Yukawa couplings (masses) and mixings are taken as inputs. The mass parameter of the Higgs is however not needed, and is a function of the other parameters (which are dominated by the top mass).

There is therefore predictive power. I defer to Thomas's talk later for the detailed analysis of the Higgs predictions situation

There is an intimate connection between the fermionic and the bosonic action (AAA,FL). Consider the fermionic action alone, a theory in which fermions move in a fixed background

The classical action is invariant for the following transformation

$$\boxed{|\Psi\rangle \to \mathrm{e}^{\frac{1}{2}\phi}|\Psi\rangle}$$

$$\boxed{D \to \mathrm{e}^{-\frac{1}{2}\phi} D \mathrm{e}^{-\frac{1}{2}\phi}}$$

Recalling the presence of $\boxed{\sqrt{\det g}}$ in the integral for the position representation of the Hilbert space it is easy to see that this is actually related to Weyl rescaling

$$\boxed{g^{\mu\nu} \to \mathrm{e}^{2\phi} g^{\mu\nu}}$$

This is a however symmetry of the classical action, not of the regularized quantum partition function

$$\boxed{Z(D) = \int [\mathrm{d}\psi][\mathrm{d}\bar{\psi}] e^{-S_\psi}}$$

and therefore there is an anomaly because a classical Weyl symmetry is not preserved at the quantum level by a regularized diffeomorphism invariant measure.

We can therefore either "correct" the action to have an invariant theory, or consider a theory in which the symmetry is explicitly broken by a physical scale

We need a scale to regularize the theory. The expression of the partition function can be formally written as a determinant, introducing a normalization dimensional constant $\boxed{\mu}$:

$$\boxed{Z(D,\mu) = \int [\mathrm{d}\psi][\mathrm{d}\bar{\psi}] e^{-S_\psi} = \det\left(\frac{D}{\mu}\right)}$$

The determinant is still infinite and we need to introduce a cutoff

The regularization can be done in several ways. In the spirit of noncommutative geometry the most natural one is a truncation of the spectrum of the Dirac operator. This was considered long ago by Andrianov, Bonora, Fujikawa, Novozhilov, Vassilevich

The cutoff is enforced considering only the first $\boxed{N}$ eigenvalues of $\boxed{D}$

Consider the projector $\boxed{P_N = \sum_{n=0}^{N} |\lambda_n\rangle \langle\lambda_n|}$ with $\boxed{\lambda_n}$ and $\boxed{|\lambda_n\rangle}$ the eigenvalues and eigenvectors of $\boxed{D}$

$\boxed{N}$ is a function of the cutoff defined as $\boxed{N = \max n \text{ such that } \lambda_n \leq \Lambda}$

We effectively use the $\boxed{N^{\text{th}}}$ eigenvalue as cutoff

The choice of a sharp cutoff could be changed in favour of a cutoff function, similar to the choice of $\boxed{\chi}$

Define the regularized partition function

$$\boxed{Z(D,\mu) = \prod_{n=1}^{N} \frac{\lambda_n}{\mu} = \det\left(\mathbb{1} - P_N + P_N \frac{D}{\mu} P_N\right)}$$

The cutoff $\boxed{\Lambda}$ can be given the physical meaning of the energy in which the effective theory has a phase transition, or at any rate an energy in which the symmetries of the theory are fundamentally different.

Under the change $\boxed{\mu \to \gamma\mu}$ the partition function changes

$$\boxed{Z(D,\mu) \to Z(D,\mu)e^{-\log\gamma \, \mathrm{tr}\, P_N}}$$

On the other side

$$\boxed{\mathrm{tr}\, P_N = N = \mathrm{tr}\, \chi\left(\frac{D}{\Lambda}\right) = S_B(\Lambda, D)}$$

for the choice of $\boxed{\chi}$ the characteristic function on the interval, a consequence of our sharp cutoff on the eigenvalues.

We found the spectral action.

We could have started without it and the renormalization flow would have provided it for free.

Let us now consider the Dirac operator for the standard model, in its barest essentiality (for our purposes). In the left-right splitting of $\boxed{\mathcal{H}}$, the operator $\boxed{D}$ it is a $\boxed{2 \times 2}$ matrix

$$\boxed{D = \begin{pmatrix} i\gamma^\mu D_\mu + \mathbb{A} & \gamma_5 S \\ \gamma_5 S^\dagger & i\gamma^\mu D_\mu + \mathbb{A} \end{pmatrix}}$$

where

$\boxed{D_\mu = \partial_\mu + \omega_\mu}$, $\boxed{\omega_\mu}$ the spin connection.

$\boxed{\mathbb{A}}$ contains all gauge fields

$\boxed{S}$ contains the Higgs field $\boxed{H}$, Yukawa couplings, mixings. . .

Technically the bosonic spectral action is a sum of residues and can be expanded in a power series in terms of Λ^{-1} as

$$S_B = \sum_n f_n \, a_n(D^2/\Lambda^2)$$

where the f_n are the momenta of χ

$$\begin{aligned} f_0 &= \int_0^\infty \mathrm{d}x \, x\chi(x) \\ f_2 &= \int_0^\infty \mathrm{d}x \, \chi(x) \\ f_{2n+4} &= (-1)^n \partial_x^n \chi(x)\Big|_{x=0} \quad n \geq 0 \end{aligned}$$

the a_n are the Seeley-de Witt coefficients which vanish for n odd. For D^2 of the form

$$D^2 = -(g^{\mu\nu}\partial_\mu\partial_\nu \mathbb{1} + \alpha^\mu \partial_\mu + \beta)$$

defining (in term of a generalized spin connection containing also the gauge fields)

$$
\begin{aligned}
\omega_\mu &= \frac{1}{2} g_{\mu\nu} \left(\alpha^\nu + g^{\sigma\rho}\Gamma^\nu_{\sigma\rho}\mathbb{1}\right) \\
\Omega_{\mu\nu} &= \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu] \\
E &= \beta - g^{\mu\nu}\left(\partial_\mu\omega_\nu + \omega_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho\right)
\end{aligned}
$$

then

$$
\begin{aligned}
a_0 &= \frac{\Lambda^4}{16\pi^2}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,\mathbb{1}_F \\
a_2 &= \frac{\Lambda^2}{16\pi^2}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\left(-\frac{R}{6} + E\right) \\
a_4 &= \frac{1}{16\pi^2}\frac{1}{360}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,(-12\nabla^\mu\nabla_\mu R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} \\
&\quad +2R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 60RE + 180E^2 + 60\nabla^\mu\nabla_\mu E + 30\Omega_{\mu\nu}\Omega^{\mu\nu})
\end{aligned}
$$

tr is the trace over the inner indices of the finite algebra $\mathcal{A}_F$ and in Ω and E are contained the gauge degrees of freedom including the gauge stress energy tensors and the Higgs, which is given by the inner fluctuations of D

We can split the partition function in the product of a term invariant for Weyl transformations, and another not invariant, which will depend on the field $\boxed{\phi}$, the dilaton.

We also need to understand what this dilaton is. We will infer its meaning from the form of the partition function.

Split the partition function in an invariant and a noninvariant part

$$\boxed{Z = Z_{inv} Z_{not}}$$

The terms in $\boxed{Z_{not}}$ exist due to the Weyl anomaly and we can calculate them. Using $\boxed{D_\phi = \mathrm{e}^{-\frac{1}{2}\phi} D \mathrm{e}^{-\frac{1}{2}\phi}}$ consider the identity

$$\boxed{Z(D) = \left(\int [d\phi] \frac{1}{Z(D_\phi)} \right)^{-1} \int [d\phi] \frac{Z(D)}{Z(D_\phi)}}$$

Since the first term is invariant by construction, $\boxed{Z_{inv} = \left(\int [d\phi] \frac{1}{Z(D_\phi)}\right)^{-1}}$, the second is the not invariant one

$$\boxed{Z_{not}(D) = \int [d\phi] e^{-S_{not}} = \int [d\phi] \frac{Z(D)}{Z(D_\phi)}}$$

This manipulation shows the meaning of $\boxed{\phi}$. We have traded in the partition function the integration over $\boxed{[d\bar{\Psi}][d\Psi]}$ with an integration over $\boxed{[d\phi]}$

The dilaton is a collective mode of fermions, and is mediating the breaking of the symmetry

We assume therefore the presence, in an earlier epoch, of a conformal point, in which the symmetry is restored. A phase in which all particles are massless, and the Higgs potential does not have the degenerate minimum

We can calculate now the bosonic action

$$\boxed{S_{not} = \ln \frac{Z(D_\phi)}{Z(D)}}$$

The calculation of $\boxed{S_{not}}$ can be done easily for $\boxed{\phi}$ constant and the result is

$$\boxed{S_{not} = \int_0^\phi \mathrm{d}t' \left(1 - \Lambda^2 \log \frac{\Lambda^2}{\mu^2} \partial_{\Lambda^2}\right) \mathrm{Tr}\, \Theta \left(1 - \frac{(e^{-\frac{t'}{2}} D e^{-\frac{t'}{2}})^2}{\Lambda^2}\right)}$$

$$\boxed{= \int_0^\phi \mathrm{d}t' \left(1 - \Lambda^2 \log \frac{\Lambda^2}{\mu^2} \partial_{\Lambda^2}\right) S_B(\Lambda,\ (e^{-\frac{t'}{2}} D e^{-\frac{t'}{2}})^2)}$$

This is a slight modification of the spectral action

Let me stress the fact that we used very few ingredients and the analysis is quite independent on the details. We have a Higgs field and a dilaton. We can therefore ask ourselves if we can say something about the effective potential involving these two fields, and its possible role in the early universe

Therefore we make the approximations of neglecting all other fields and the derivative of the Higgs, and retain in the heat kernel expansion only the terms involving the Higgs field $\boxed{H}$ and the dilaton $\boxed{\phi}$

In this view, unlike earlier work, we do not consider a formal RG flow, but the time evolution of the system, fixing the normalization by the request that the vacuum energy in the present epoch is vanishing small, and using the conformal point as boundary condition

The behaviour of $\boxed{D}$ under Weyl rescaling gives the transformation of $\boxed{H}$ under such transformation. Only the $\boxed{H^4}$ term in the effective potential is invariant, and it can be multiplied by a constant quantity ($\boxed{\phi_0}$). This gives, in this approximation, the invariant part of the effective potential

The other terms of the effective potential can be calculated using the heat kernel. The effective potential, sum of the invariant and not invariant part has the form has the form

$$\boxed{V = V_0 + a(e^{2\phi} - 1) + bH^2(e^{2\phi} - 1) - cH^4(\phi + \phi_0) + EH^2}$$

The coefficients are in principle calculable at one loop, and are functions of the the parameters $\boxed{\Lambda}$ and $\boxed{\mu}$ and there is another (integration) constant $\boxed{\phi_0}$, in principle also calculable.

with a shift $\boxed{\phi \to \phi - \phi_0}$ and a redefinition if the constants the potential can be written as

$$\boxed{V = V_0 + Ae^{4\phi} + BH^2e^{2\phi} - CH^4 + EH^2}$$

The normalization constant $\boxed{\mu}$ can be fixed requiring that the constant term in the action proportional to $\boxed{\Lambda^4}$ vanishes. This give

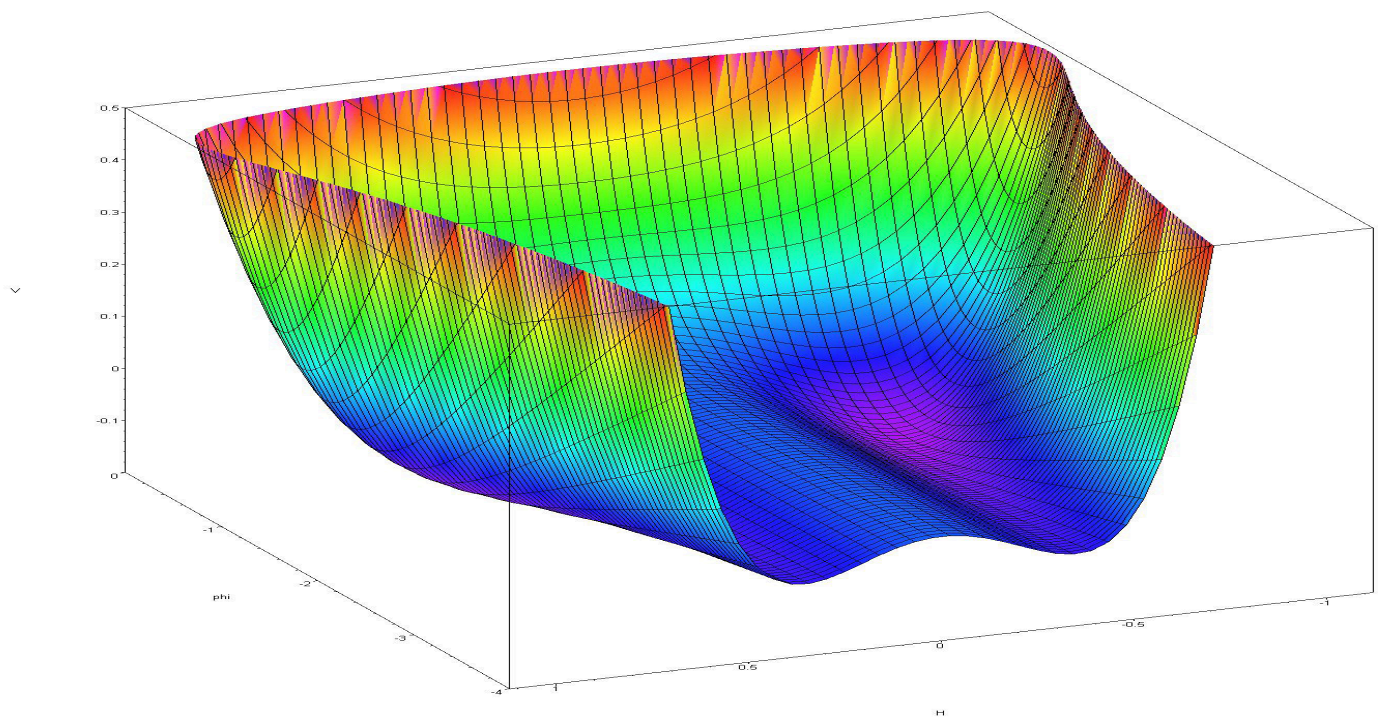
$$\boxed{\frac{\Lambda}{\mu} = e^{\frac{1}{4}}}$$

One can then evolve the potential and finds the following properties for the Higgs-dilaton potential

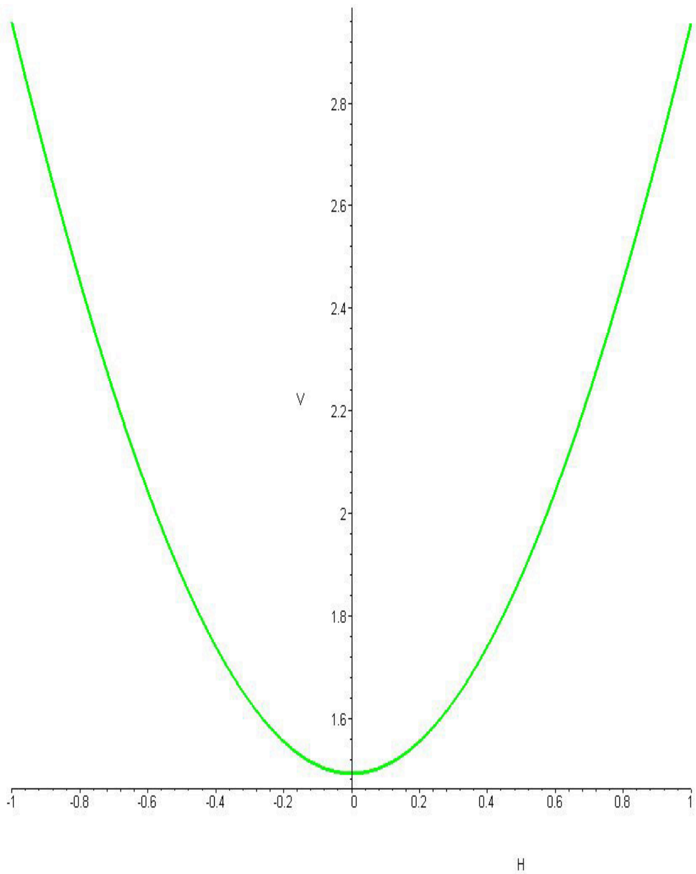
- The existence of a local minimum

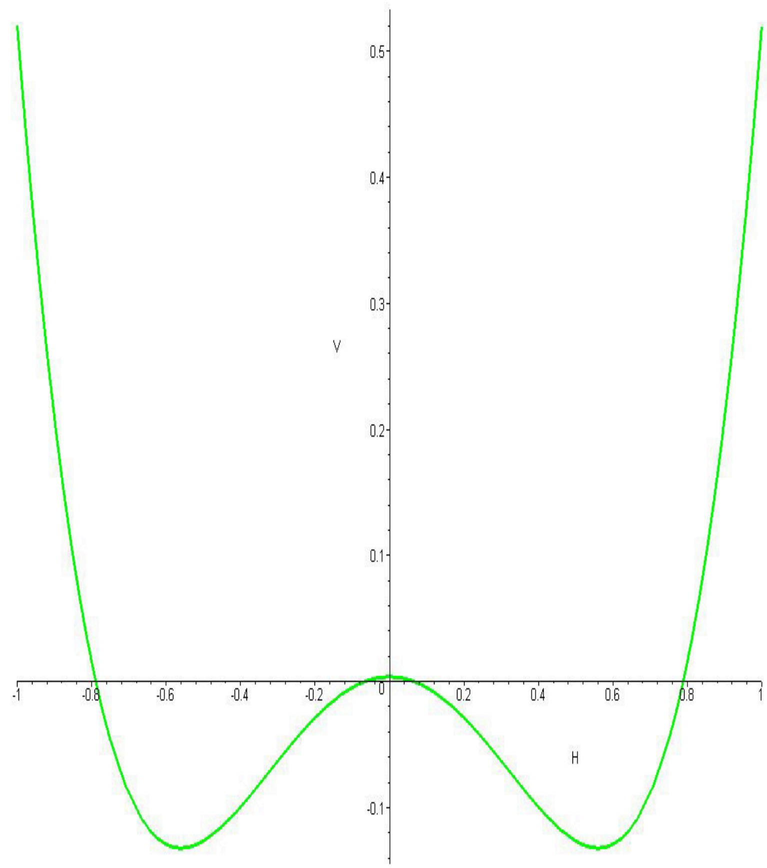
- The existence of an unbroken phase from which the potential may roll down to the broken phase

Plot of the effective Higgs-Dilaton potential:



We see that for different values of $\boxed{\phi}$, the potential $\boxed{V(H)}$ has a transition from a symmetric to a broken phase.





What I have describe so far referred to the case of a constant dilaton.

It is possible to actually calculate explicitly the action of the collective modes:

$$\boxed{S_{coll} \equiv \int d^4x\sqrt{g}\left(A\left(e^{4\phi}-1\right)+BH^2\left(e^{2\phi}-1\right)-C\phi H^4-\alpha_1\left(e^{2\phi}-1\right)R+\alpha_2 e^{2\phi}\left(\phi_{;\mu}\phi_{;}{}^{\mu}\right)\right.}$$

$$\boxed{\left.-\alpha_3\ \phi\left(3y^2\left(D_\mu H D^\mu H-\tfrac{1}{6}RH^2\right)+G^i_{\mu\nu}G^{\mu\nu i}+W^\alpha_{\mu\nu}W^{\mu\nu\alpha}+\tfrac{5}{3}B_{\mu\nu}B^{\mu\nu}-\tfrac{9}{16}C_{\mu\nu\rho\lambda}C^{\mu\nu\rho\lambda}\right)\right\}}$$

$$\boxed{-\alpha_4\left(12R\left(\phi_{;\mu}^{\ \ ;\mu}+\phi_{;\mu}\phi_{;}{}^{\mu}\right)+11\phi G_B+44G^{\mu\nu}\phi_{;\mu}\phi_{;\nu}+14\left(\phi_{;\mu}^{\ \ ;\mu}+\phi_{;\mu}\phi_{;}{}^{\mu}\right)^2+22\left(\phi_{;\mu}^{\ \ ;\mu}\right)^2\right)\Big)}$$

where $\boxed{G_{\mu\nu}}$ stands for the Einstein tensor and the constants $\boxed{A, B, C, \alpha_1..\alpha_4}$, are defined as follows:

$$A = \left(2\log\frac{\Lambda^2}{\mu^2}-1\right)\frac{45\Lambda^4}{32\pi^2}, \quad B=\left(1-\log\frac{\Lambda^2}{\mu^2}\right)\frac{15\Lambda^2 y^2}{16\pi^2}, \quad C=\frac{3z^2}{4\pi^2},$$

$$\alpha_1 = \left(1-\log\frac{\Lambda^2}{\mu^2}\right)\frac{15\Lambda^2}{32\pi^2}, \quad \alpha_2=\left(1-\log\frac{\Lambda^2}{\mu^2}\right)\frac{45\Lambda^2}{16\pi^2}, \quad \alpha_3=\frac{1}{4\pi^2}, \quad \alpha_4=\frac{1}{128\pi^2}.$$

Conclusions

- Starting from the general framework of noncommutative geometry it is possible to say something of phenomenological interest. These models are quite rigid. Not all Yang-Mills theories come from NCG for example.

- Although the models are not yet fully ready to be confronted with experiment, they are promising

- Weyl symmetry (and anomaly) play an important role in the genesis of the spectral action

- The effective Higgs-dilaton potential also emerges with desirable features: broken and symmetrical phases, roll down

- It would be interesting to investigate the full predictive power of these models.