Noncommutative Geometry for Quantum Spacetime

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In these talks I will discuss some generalizations of the geometry of spacetime and how they may be relevant when the gravitational effects become important, such as the early universe

In the first talk I will attempt a general review, from a personal point of view, of the various attempts which generically go under the name *"noncommutative geometry"*

In the next talk I will then describe the spectral approach to geometry and field theory

I then hope than in tomorrow's discussion there will be a comparison with other proposed "geometries"

There are several reasons for which it seems unreasonable to assume that geometry of spacetime at the very beginning of the universe should be the same we use at this late stage of the universe,low temperature and low energy.

We are used to changes in geometry. Special relativity makes use of non-Euclidean geometry and introduced the generalization from space to spacetime. General relativity needs non flat geometry. Yet both these theory use some "classical" geometry, based on the concept of *event* which is basically the concept *point* of spacetime

The biggest changes come from quantum mechancis

A quick (and not really correct) way to see that in quantum mechanics the concept of point (of phase space) is not valid is given by the Heisenberg Microscope

The idea is that to "see" something small, of size of the order of Δx, we have to send a "small" photon, that is a photon with a small wavelength λ, but a small wavelength means a large momentum $p = h/\lambda$. In the collision there will a transfer of momentum, so that we can capture the photon. The amount of momentum transferred is uncertain.

In quantum mechanics a point in phase space is an untenable concept because of the Heisenberg uncertainty principle:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

The "not correct" comment comes from the fact that I need used relativity. But while there is a consistent non relativistic point particle quantum mechanics, for the relativistic case we only have a quantum field theory.

We know what has happened. The observables, which in classical mechanics are commutative functions on the phase space, have become noncommutative operators on the Hilbert space of wave functions

A (pure) state in classical mechanics is a point of phase space, an observable is something which gives a real number for each state (the value of the function on the point)

A (pure) state in quantum mechanics is a vector on the Hilbert space, an observable is something which gives a real number for each state (the expectation value of the operator)

The difference is noncommutativity

There is an important mathematical result: The information on the (classical) phase space is encoded in the (commutative) algebra of observables, i.e. in the functions on the space

A theorem (Gelfand-Naimark) shows a complete equivalence between commutative C^*-algebras and Hausdorff topological spaces

A Hausdorff space is one for which points are separable. A C^*-algebra is an associative algebra with a norm and a complex conjugation

Given an Hausdorff space it is always possible to construct a commutative C^*-algebra: continuous complex valued functions.

The converse is also true, an arbitrary C^*-algebra is always the algebra of continuous complex valued functions on some Hausdorff space. The points of the space are the pure states of the algebra, the topology is given by convergence.

This duality has led some people, starting with Von Neumann, but principally Alain Connes, to an attempt to transcribe all properties of ordinary spaces in algebraic terms

Thus the emphasis in the description of geometry switches from points to fields

The topology is encoded by the algebra, which can always be represented as operators on some Hilbert space (loosely speaking, every algebra is a matrix algebra, possibly infinite dimensional)

The metric structure is encoded in a (generalized) Dirac operator D, which "knows" about the metric, and is used to build the differential calculus (forms). Integrals become traces of operators with the inverse of D playing the role of the measure

What if the algebra is noncommutative? In this case we have a **Noncommutative Space**

If the algebra is noncommutative the identification of points with pure states (or irreducible representations) fails. Often the Hausdorff topology gives a single points

Nevertheless the topological information about the space is encoded in the noncommutative algebra. This algebra can always be represented as operators on an Hilbert space, and further geometrical properties, such as the metric, can be encoded in the generalized Dirac operator D operator

If we succeed in transcribing objects of ordinary geometry in algebraic terms, then the generalization is "simply" done just assuming that the algebra is noncommutative

More on this later in the second talk, and in Thomas Schucker's talks next week

The noncommutative structure of spacetime

So far we have been discussing the noncommutativity of phase space. In quantum mechanics however configuration space is still an ordinary space. So it is in general relativity

Is it legitimate to expect the usual geometry to hold to all scales?

As far as I know the first to consider the possibility of noncommutative spacetime was Heisenberg who, in a letter to Peierls in 1930, expressed the hope that a noncommutative structure might eliminate the infinities of quantum field theory, which at that time were considered a problem.

In 1935 Bronstein observed that the combination of general relativity and quantum mechanics might create problems for the localization of events.

At the same caricature level I used before for the Heisenberg microscope:

In order to “measure” the position of an object, and hence the “point” in space, one has use a very small probe, and quantum mechanics forces us to have it very energetic, but on the other side general relativity tells us that if too much energy is concentrated in a region a black hole is formed.

This is the region in which the theory to use is Quantum Gravity. Unfortunately a theory we do not yet have

In fact the two problems are related. A quantum gravity theory needs spacetime to be a different object from the one used in classical geometry

A solution of the localization problem is given by spacetime with noncommuting coordinates, whose commutator is a central operator. This was introduced by Doplicher-Fredenhagen-Roberts. I think Gherardo Piacitelli talked about it.

In tomorrow's discussion I would like to stimulate a comparison among the various geometries avalaible now, for example the one suggested by loop quantum gravity, or strings.

Whatever theory we make we must bear in mind the in some limit we should be able to find again the classical space and its structures. This suggests to use a method developed for quantum mechanics:

Deformation of spaces

This theory was developped for quantum mechanics.

Take the algebra of classical observables, functions multiplied with the commutative product, and introduce a deformed (Gronewöld-Moyal) product:

$$(f \star g)(x,p) = f e^{\frac{i\hbar}{2}\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x}} g = fg + \frac{i\hbar}{2}(\partial_x f \partial_p g - \partial_p f \partial_x g) + O(\hbar^2)$$

So that to first order in $\hbar$

$$f \star g - g \star f = i\hbar\{f,g\} + O(\hbar^2)$$

This is a concrete realization of Dirac's correspondence principle.

The commutator is a deformation of the Poisson bracket, in the limit $\hbar \to 0$ one finds again the classical structure

This is a way to describe quantum mechanics as a deformation of classical mechanics

The usual phase is still there, but the functions defined on it, the observables form now a noncommutative algebra

It is possible to consider different deformed products, for example one which reproduces normal ordered products of operators. They correspond to different quantization schemes

In the spirit of what I said before one can threat a noncommuting space deforming the algebra of functions with a $\star$ product similar to the one introduced in quantum mechanics, with $\hbar$ replaced by an antisymmetric matrix θ:

$$f \star g = f e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}} g$$

In this way we encode the noncommutativity of spacetime in the deformation of the algebra

This product appeared in the celebrated Witten's article on non-commutative geometry and strings, thus fostering an enormous surge of interest in the topic.

Noncommutative Field Theory

Deform of a commutative theory with the presence of a star product among the fields. For example

$$S = \int d^d x \partial_\mu \varphi \star \partial^\mu \varphi + m^2 \varphi \star \varphi + \frac{g^2}{4!} \varphi \star \varphi \star \varphi \star \varphi$$

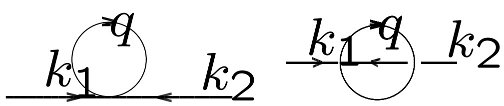
For the Grönewold-Moyal product the $\star$ on the first two terms is redundant because $\int d^d x f \star g = \int d^d x f g$

What physics comes out of these theories?

The free theory is unchanged because of the integral property. But the vertex gets a phase. For the example $\varphi^{\star 4}$:

$$V = (2\pi)^4 g \delta^4 \left(\sum_{a=1}^{4} k_a \right) \prod_{a<b} \mathrm{e}^{-\frac{i}{2}\theta^{\mu\nu} k_{a\mu} k_{b\nu}}$$

The vertex is not anymore invariant for exchange of the momenta (only for cyclic permutations), and causes a difference between planar and nonplanar diagrams



A consequence of this is Ultraviolet/Infrared Mixing Minwalla-Seiberg-Van Raamnsdong. The phenomenon for which some ultraviolet divergences disappear, just to reappear as infrared divergences

Should we take seriously the fact that the world is really described by this kind on theory? Should we study in detail its consequences?

The answers to these two question is, in my opinion, no and yes.

It is a full fledged theory which uses a geometry of spacetime different form the usual one, and it uses a well known technology and as such it can give us useful information

The fact that this product uses a know technology is both its strength and its weakness. In quantum mechanics position and momenta are intrinsically different,and there is no symmetry which should connect them, except in particular case

Spacetime is a theory which should threat all directions in space on the same footing among them and with time. What do we do with Lorents symmetry?

Let me notice first a few well known facts.

In the scheme that I am preparing θ must be a constant quantity if we want the product to be associative.

Associativity is necessary for gauge theories for example, the Lagrangian, under a gauge transformation $F^{\mu\nu}F_{\mu\nu} \to UF^{\mu\nu}F_{\mu\nu}U^\dagger$, and what is invariant is the action $\int dx F^{\mu\nu}F_{\mu\nu}$

Setting a $\theta(x)$ will ruin associativity

Finding associative deformations is a difficult task, Kontsevich has won a Field medal for it!

If were to take seriously the fact that the world is described by this kind of noncommutative field theory which are the consequences? How do we measure $\theta^{\mu\nu}$, a quantity of the order of ℓ_P^2?

At this level $\theta^{\mu\nu}$ is a background quantity, which selects two directions in space (analog of electric and magnetic fields). This breaks Lorentz invariance and the noncommutativity will have left its imprinting in the early universe

Direct accelerator measurements are more difficult because the earth rotation washes up the effect. But one can look for otherwise forbidden processes

With Mangano, Miele and Peloso I wrote ten years ago one of the first papers on applications of the Moyal product to the early universe. The observational signatures where coming from the breaking of Lorentz symmetry,

It is not easy to distinguish predictions coming from these kind of theories from other breakings of Lorentz invariance

A quantum spacetime may require a quantum symmetry?

A deformation of spacetime may require a deformation of symmetries. The tool may be quantum Groups and Hopf Algebras

A Lie group is a manifold, and therefore it is a topological space, described by its commutative algebra of functions. It has however added structure: it makes sense to multiply "points", there is an identity, an inverse of every point.

This structure is encoded in the algebra of functions as a co-product, which from a function of one variable gives a function of two variables:

$$\Delta(f)(g_1, g_2) = f(g_1 g_2)$$

The Lie algebra level (infinitesimal transformations, or differential operators), the coproduct in the group induces a coproduct in the algebra

$$\Delta(L) = L \otimes I + I \otimes L$$

Which is the Leibnitz rule when L, element of the Lie algebra is seen as a differential operator. This (and other structures) gives the structure of a Hopf algebra

A quantum group is what we obtain when the algebra of functions on the group becomes noncommutative. It is then necessary to deform commutation relations and/or coproducts.

The Hopf algebra which is relevant in this context is θ-Poincaré

This is a deformation induced by a Drinfeld twist

$$\mathcal{F} = e^{-\frac{\mathrm{i}\,\theta^{ij}}{2}\partial_i \otimes \partial_j}$$

with this twist deform the usual product

$$m_0 : f \otimes g \longrightarrow f \cdot g$$

into the Moyal product

$$m_\star = m_0 \mathcal{F}^{-1} : f \otimes g \longrightarrow f \star g$$

I have no time to go into the connections of this theory with braiding, Yang-Baxter and other beautiful mathematical structures

Consider the symmetry to be a twisted quantum symmetry (Drinfeld, Wess and the Münich group: Aschieri, Blohmann, Dimitrieviċ, Meyer, Schupp, Chaichian-Kulish-Nishijima-Tureanu, Oeckl, Majid . . .)

Consider the usual action of the Lie algebra L of differential operators on the algebra $\mathcal{A}$ of functions with the usual commutative product

The usual product can be seen as a map from $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$

$$m_0(f \otimes g) = fg$$

with pointwise multiplication

The Leibnitz rule imposes a coalgebra structure of the Lie algebra:

$$\ell(fg) = \ell(f)g + f\ell(g) = m_0(\Delta(L)(f \otimes g))$$

where ℓ is a generic first order differential operator

$$\Delta : L \to L \otimes L$$

$$\Delta(\ell) = \ell \otimes 1 + 1 \otimes \ell$$

The coproduct tells how to put together representations, and how an operator acts on two copies of the module.

Consider the Moyal product as follows

$$(f \star g)(x) = m_0[\mathcal{F}^{-1} f \otimes g] \equiv m_\theta[f \otimes g]$$

where $m_0(f \otimes g) = fg$

is the ordinary product and

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_{x_\mu} \otimes \partial_{y_\nu}} = e^{-\frac{i}{2}\theta(\partial_{x_0} \otimes \partial_{y_1} - \partial_{x_1} \otimes \partial_{y_0})}$$

is called the twist.

The noncommutative product is obtained first twisting the tensor product, and then using the ordinary product.

With the twist we have to revise the Leibnitz rule:

$$\partial_\mu(f \star g) = m_\theta \Delta_\theta(f \otimes g) = m_0 \Delta(\partial_m u)(\mathcal{F}^{-1}(f \otimes g))$$

where

$$\Delta_\theta = \mathcal{F} \Delta \mathcal{F}^{-1}$$

The algebra structure remains unchanged, what changes is the coalgebra structure, that is the way to "put together representations".

counit and antipode remain unchanged.

We have this deformed the coalgebra structure of the Poincaré Lie algebra. In particular:

The Lie algebra structure (commutators) is not changed. What changes is the coalgebra, at the level of the Lorentz group

$$\Delta_{\mathcal{F}}(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu$$

$$\Delta_{\mathcal{F}}(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta}\left((\eta_{\alpha\mu}P_\nu - \eta\alpha\nu P_\mu) \otimes P_\beta + P_\beta\left(\eta_{\beta\mu}P_\nu - \eta\beta\nu P_\mu\right)\right)$$

The fact that the algebra is the same means that we can still use the casimirs and the representations of the usual algebra, with thus concepts of mass, spin etc.

The twisted framework for noncommutative field theory i is not free from controversies

We have changed the tensor product, and therefore one should twist all products in an appropriate way. This shows for example that taking other products, related to other quantization schemes, gives the same "physics" at the level of the S-matrix.

Using this scheme there are modifications of gravity Wess et al, Aschieri, Castellani . . .

We are probably still lacking a "canonical" procedure to understand the twist.

In particular we do not have a "measurement" theory in the case of twisted symmetries

My point of view is that one has to *twist all the way*, namely apply the twist operator to all tensor products one encounter. In this way for example one can show that the S-matrix in scattering processes depends on the quantization scheme, and not on the equivalent products one uses to represent it

If two observers wish to compare the result of their observation, they have to implicitly use the tensor product of their respective Hilbert space and observables.

Measurement issue are a very complicated and controversial issue. So here I will just "throw the stone and hide my hand"...

Another possibility could be κ-Minkowki. This is the homogenous space of the κ-Poincaré quantum group, and it is characterized by the commutation relations

$$[x_i, x_0] = i\lambda x_i, \quad [x_i, x_j] = 0$$

The commutation relations for κ-Poincaré are:

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[M_i, P_j] &= i\epsilon_{ijk} P_k \\
[M_i, P_0] &= 0 \\
[N_i, P_j] &= -i\delta_{ij}\left(\frac{1}{2\lambda}(1 - e^{2\lambda P_0}) + \frac{\lambda}{2} P^2\right) + i\lambda P_i P_j \\
[N_i, P_0] &= iP_i \\
[M_i, M_j] &= i\epsilon_{ijk} M_k \\
[M_i, N_j] &= i\epsilon_{ijk} N_k \\
[N_i, N_j] &= -i\epsilon_{ijk} M_k
\end{aligned}
$$

All these commutation relations become the standard ones for $\lambda \to 0$. The bicrossproduct basis is peculiar as κ-Poincaré acts *covariantly* on a space that is necessarily deformed and noncommutative. This is a consequence of the non cocommutativity of the coproduct which, always in the bicrossproduct basis, reads:

$$\begin{aligned}
\Delta P_0 &= P_0 \otimes 1 + 1 \otimes P_0 \\
\Delta M_i &= M_i \otimes 1 + 1 \otimes M_i \\
\Delta P_i &= P_i \otimes 1 + e^{\lambda P_0} \otimes P_i \\
\Delta N_i &= N_i \otimes 1 + e^{+\lambda P_0} \otimes N_i + \lambda \varepsilon_{ijk} P_j \otimes M_k
\end{aligned}$$

The Casimir of this quantum group provide a deformation of the Energy-Momentum dispersion relation and this could be used to explain a vaiety of things. The problem is that, being the commutation relations nonlinear, nonlinear changes of coordinates are allowed, and therefore these dispersion relations become basis-dependent.

I will not dwell more on the issue of κ-Minkowski. Some attempts to connect it with the real world have been made, especially in connection with Doubly special relativity.

I am convinced that, while the geometry of quantum gravity has to be new, it is likely that we have not yet hit on the correct one. Apart form the ones described here there are more approach, which we may discuss in tomorrow's session

Wether the attempts i described earlier are getting close to the "real" one I sincerely do not know.

They teach something about the structure of field theory, and as such are certainly important.

In the second talk I will take a "bottom-up" approach. Sustained by a strong mathematical framework, I will try to start form the physics we know, that of the standard model, and work my way up.