The Hopf way to Planckland

without breaking Poincaré and gauge symmetries

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Long before getting to Planckland field theory has a problem:

Ultraviolet Divergences

In the perturbative expansions it is necessary to integrate over the loops, and often the resulting integral diverges in the limit in which the internal momentum goes to infinity

The solution is known: The renormalization programme

Very roughly speaking the renormalization programme goes as follows:

- The presence of a dimensional cutoff regularizes the theory

- The infinities are subtracted, giving meaning to the remaining finite quantities

- The behaviour of physical quantities at different scales is given by the renormalization group

The cutoff may be a conventional scale, that one should send to infinity at the end, or it may have physical meaning

In this case the theory is in reality an effective theory, valid only below the energy cutoff scale

We are all convinced (at least in this conference) that at the Planck scale something will happen. And probably a noncommutative geometry will set. Bronstein, Doplicher

One coud entertain the hope that the presence of a theory with a fundamental scale could regularize a field theory.

In fact at the beginning of the study of "noncommutative field theories" based on the Grönewold-Moyal product this was precisely the hope.

Let me express the product as a twisted convolution of Fourier transforms

$$(\tilde{f} \star \tilde{g}) + \frac{1}{(2\pi)^2} \int dq \tilde{f}(p-q)\tilde{g}(q) e^{ip_\mu \theta^{\mu\nu} q_\nu}$$

$\theta^{\mu\nu}$ ia a dimensional quantity with the dimension of a length squared

Unfortunately this kind of noncommutativity does not eliminate all ultraviolet infinities

Worse! Some (but not all) of the diverging terms are regularized in the ultraviolet, but develop an infrared divergence

This the Ultraviolet/Infrared mixing Minwalla-Van Raamsdonk-Seiberg

One could try to modify the product hoping to have a more decisive cutoff. And use for example the Wick-Voros product, which in two dimension with $z = z_1 + ix_2$ reads

$$f \star g = \sum_n \left(\frac{\theta^n}{n!}\right) \partial_+^n f \partial_-^n g = f \mathrm{e}^{\theta \overleftarrow{\partial_+} \overrightarrow{\partial_-}} g$$

and in momentum space

$$(\tilde{f} \star \tilde{g})(p) = \frac{1}{(2\pi)^2} \int dq \tilde{f}(p-q)\tilde{g}(q) e^{p_\mu \theta^{\mu\nu} q_\nu} e^{p_\mu q_\mu}$$

Doing a field theory with this product one finds as propagator

$$G_{0_V}^{(2)}(p) = \frac{\mathrm{e}^{-\frac{\theta}{2}|\vec{p}|^2}}{p^2 - m^2}$$

This looks like a cutoff, and gives some hope that a resulting theory with this product may be convergent

Unfortunately the hope is short lived: in the integration for the internal momenta the extra factors simplify and the divergences come out to be the same as in the Moyal case Galluccio-FL-Vitale

This is to be expected since the two schemes correspond to equivalent "quantization procedures"

The proof that at the level of the S-matrix the two theories are equivalent uses the Hopf algebra structure.

These noncommutative products break Lorentz invariance and retain only translational invariance

They retain however a quantum symmetry given by the Hopf algebra invariance. More on this later

General translation invariant star products have been studied. Galluccio-FL-Vitale, Ardalan-Saagoghi. The expression in omentum space is:

$$(\tilde{f} \star \tilde{g})(p) = \frac{1}{(2\pi)^2} \int dq \tilde{f}(p-q)\tilde{g}(q)\mathcal{K}(p,q)$$

with

$$\mathcal{K}(p,q) = H^{-1}(p)H(q)H(p-q)\mathrm{e}^{\mathrm{i}\,\alpha(p,q)}$$

$H(q)$ an arbitrary even real function and α constrained by associativity

$$\alpha(p,q) = \theta^{\mu\nu} p_\mu q_\nu + \partial\beta(p,q) = \theta^{\mu\nu} p_\mu q_\nu + \beta(q) - \beta(p) + \beta(p-q)$$

where I have emphasized the fact that it is defined up to a coboundary term, $\partial\beta(p,q)$. β is a real odd function.

$\theta^{\mu\nu}$ is antisymmetric and constant, responsible for the noncommutativity of the product. If $\theta = 0$ the product is commutative. Commutative products are associated to coboundaries.

Lorentz invariance further constrains the form of the product. Only H survives, since θ not invariant and β is an odd function of the modulus of momenta.

Full Poincaré invariance forces upon us the commutativity of the product

We stress however that this does not mean that the product is the usual pointwise one.

Ideally one would like some action which implements a strict cutoff on the momenta. Without breaking Lorentz invariance, nor gauge invariance. Something like:

$$S_{\mathrm{QED}} = \int_\Lambda \frac{\mathrm{d}^4 p}{(2\pi)^4} \Bigg\{ \bar{\tilde{\psi}}(-p)(\gamma_\mu p^\mu + m)\tilde{\psi}(p)$$

$$+ e\bar{\tilde{\psi}}(-p) \int_\Lambda \frac{\mathrm{d}^4 q}{(2\pi)^4} \Theta_\Lambda(p-q)\gamma^\mu \tilde{A}_\mu(p-q)\tilde{\psi}(q)$$

$$+ \frac{1}{4}\left(\tilde{F}_{\mu\nu}(-p)\tilde{F}^{\mu\nu}(p) + \frac{1}{2\xi}(p_\mu \tilde{A}^\mu(p))^2 \right) \Bigg\}$$

Here $\int_\Lambda \mathrm{d}^4 p = \int_0^\Lambda p^3 \mathrm{d}p$ (we are in the Euclidean case), and Θ is the characteristic function of the four-sphere of radius Λ

$$\Theta_\Lambda(p) = \begin{cases} 1 & p^2 < \Lambda \\ 0 & p^2 \geq \Lambda \end{cases}$$

Polchinski introduced a momentum cutoff function to find an exact renormalization group equation for a scalar theory. The procedure is of not easy applicability to gauge theories, and there are problems for nonabelian Yang-Mills gauge theories.

The idea is to see if a deformed product can implement the cutoff and preserve the symmetries of the theory Ardalan-Arfaei-Ghasemkhani-Sadooghi, FL-Vitale

This can be done, with a commutative star product:

$$(f \star h)(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{H(p)} e^{\,\mathrm{i}\, px} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} [H(p-q)\tilde{f}(p-q)][H(q)\tilde{h}(q)]$$

We will consider H to be a cutoff function, that is, $H(0) = 1$, $H(p) \simeq 1$ for $p^2 < \Lambda^2$, and thereafter $H \to 0$ rapidly

The presence of H in the denominator of the product prevents it from vanishing, and in particular to be a sharp cutoff, identically vanishing after some scale

it is easy to see that

$$\int \mathrm{d}^4x (f \star h)(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} H^2(p) \tilde{f}(-p) \tilde{h}(p)$$

$$f \star 1 = 1 \star f = f$$

The new product belongs to the same cohomology class as the point-wise product. For each invertible H we have an isomorphism of algebras

$$\varphi : (\mathcal{A}_\star, \mu_\star) \to (\mathcal{A}_0, \mu_0)$$

A non vanishing cut-off function provides essentially a field redefinition

$$\widetilde{\varphi(f)}(p) = H(p)\tilde{f}(p)$$

Despite these warnings we will show that the regularized theory with the sharp cutoff can be properly defined as the limit $H(p) \to \Theta_\Lambda(p)$ of well defined theories with analytic cutoff.

The Iranian group has applied this deformed product to gauge theory and proven a Ward identity

In fact the theory enjoys a deformed $U(1)$ Hopf symmetry

Ordinary gauge theories with group of gauge transformations gauge $\widehat{G}$ are modified replacing the point-wise product with a non-local. The resulting field theories are invariant under the deformed gauge transformations

$$\phi(x) \longrightarrow g_\star(x) \triangleright_\star \phi(x) = \exp_\star\left(i\alpha^i(x)T_i\right) \triangleright_\star \phi(x)$$

$\triangleright_\star$ indicates the action, for the nonabelian case includes a matrix multiplication, T_i are the Lie algebra generators, $g_\star(x)$ are defined as star exponentials

$$g_\star(x) = \exp_\star\left(i\alpha(x)^i T_i\right) = 1 + i\alpha^i(x)T_i - \frac{1}{2}(\alpha^i \star \alpha^j)(x)T_iT_j + \dots$$

At the infinitesimal level we have

$$\phi(x) \longrightarrow \phi(x) + i(\alpha \triangleright_\star \phi)(x) = \phi(x) i \left(\alpha^j(x) \star (T_j \triangleright \phi) \right)(x)$$

The deformed Lie multiplication reads $[\alpha, \tilde{\alpha}]_\star(x) = (\alpha \star \tilde{\alpha})(x) - (\tilde{\alpha} \star \alpha)(x)$

Note that in noncommutative field theory there is a problem since

$$(\alpha \star \tilde{\alpha})(x) - (\tilde{\alpha} \star \alpha)(x) = \left((\alpha^i \star \tilde{\alpha}^j)(x) + (\tilde{\alpha}^j \star \alpha^i)(x) \right) [T_i, T_j]$$

$$+ \left((\alpha^i \star \tilde{\alpha}^j)(x) - (\tilde{\alpha}^j \star \alpha^i)(x) \right) \{T_i, T_j\}$$

which only closes for the group $U(N)$ in the adjoint and fundamental representations. The problem is solved for example in twisted gauge theories. In the present case the definition is perfectly viable for any Lie group, because the product is commutative, therefore the term proportional to the anticommutator $\{T_i, T_j\}$ vanishes.

For the action of the group we have to take into account the whole Hopf algebra to describe the symmetry. Even if in this $U(1)$ case the Lie algebra is trivial, still the Hopf algebra structure is nontrivial

Generalizing the Leibnitz rule we have

$$\alpha \triangleright (f \star h)(x) = \mu_\star \circ \Delta_\star(\alpha)(f \otimes h)(x)$$

coming from the cocommutative and coassociative coproduct

$$\Delta_\star(\alpha)(f \otimes g) = (\alpha_\star \otimes \mathrm{id} + \mathrm{id} \otimes \alpha_\star)\,(f \otimes g) = \alpha \triangleright_\star f \otimes g + f \otimes \alpha \triangleright_\star g$$

with $(\alpha \triangleright_\star f)(x) = (\alpha^i \star T_i \triangleright f)(x)$, where the action of the ungauged Lie algebra in the appropriate representation is not modified.

Antipode e counits are $(S_\star(\alpha) \triangleright f)\,(x) = -(\alpha \triangleright_\star f)(x)$ and $\epsilon_\star(\alpha) = \epsilon(\alpha) = 0.$

We have the following action

$$S_H = \int \mathrm{d}^4x \left\{ -\bar{\psi} \star (i\gamma^\mu \partial_\mu - m)\psi + e\bar{\psi} \star \gamma^\mu A_\mu \star \psi + \frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \frac{1}{2\xi}(\partial_\mu A^\mu)^{\star 2} \right\}$$

The deformed gauge transformations for matter and gauge fields explicitly read

$$\psi(x) \to \psi(x) + ie(\alpha \star \psi)(x), \quad \bar{\psi}(x) \to \bar{\psi}(x) - ie(\alpha \star \bar{\psi})(x), \quad A_\mu(x) \to A_\mu($$

in momentum space

$$\delta\tilde{\psi}(p) = \mathrm{i}\, eH^{-1}(p) \int \frac{\mathrm{d}^4q}{(2\pi)^4} H(p-q)\tilde{\alpha}(p-q)H(q)\tilde{\psi}(q)$$

$$\delta\tilde{\bar{\psi}}(p) = -\mathrm{i}\, eH^{-1}(p) \int \frac{\mathrm{d}^4q}{(2\pi)^4} H(p-q)\tilde{\alpha}(p-q)H(q)\tilde{\bar{\psi}}(q)$$

$$\delta\tilde{A}_\mu(p) = \mathrm{i}\, p_\mu \tilde{\alpha}(p)$$

On using the expression for the integral of deformed product the action is

$$S_H = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \Bigg\{ H(p)\tilde{\bar{\psi}}(-p)(\gamma_\mu p^\mu + m)H(p)\tilde{\psi}(p)$$

$$+eH(p)\tilde{\bar{\psi}}(-p) \int \frac{\mathrm{d}^4 q}{(2\pi)^4} H(p-q)\gamma^\mu \tilde{A}_\mu(p-q)H(q)\tilde{\psi}(q)$$

$$+\frac{1}{4}H^2(p)\left(\tilde{F}_{\mu\nu}(-p)\tilde{F}^{\mu\nu}(p) + \frac{1}{2\xi}(p_\mu \tilde{A}^\mu(p))^2\right)\Bigg\}$$

Although the modified product contains and inverse function of H, this has disappeared from the action

The effect of the product is the modification of the fermion and photon propagators

$$S(p) = \frac{H^2(p)}{\gamma^\mu p_\mu + m} \quad G_{\mu\nu}(p) = -\,\mathrm{i}\,\frac{H^2(p)}{p^2}\left(\delta_{\mu\nu} + (\xi - 1)\frac{p_\mu p_\nu}{p^2}\right)$$

The three point function at tree level inherits the factors form the cubic term of the action, and therefore is itself regulated

$$G(p,q,k) = \gamma^\mu H(p)H(q)H(k)\delta(p-q-k)$$

where p, q, k are the momenta of an incoming electron and outgoing photon and electron respectively.

It is gauge invariant by construction (with respect to the deformed symmetry discussed above), but it is also possible to prove the relevant Ward identities, the derivation mirrors the usual one

For the product to be defined it is necessary that the function H do not vanish anywhere. But then the deformation is isomorphic to the point-wise one. The action however can be defined for arbitrary cutoff functions, including those which identically vanish for p^2 larger than some scale.

We thus consider a sequence of analytic cutoff functions which converge to the sharp cutoff $\Theta_\Lambda(p)$

$$H_\epsilon(p) \to \Theta_\Lambda(p)$$

A possible choice is for example the following sequence of functions

$$H_\epsilon(p) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{p^2-\Lambda^2}{\epsilon\Lambda^2}\right)$$

At each stage of the limiting procedure (to be understood in the weak, nonuniform sense) the action preserves the symmetries, while converging to the cutoff-action introduced at the beginning of this talk.

The theory with the sharp cutoff cannot be defined with a deformed product, nevertheless, being a limit of Hopf-gauge invariant theories, it enjoys their symmetries, and the proof of the Ward identities still goes through.

there is a definition of equivalence of Hopf algebras. There must exist a map

$$\varphi : \mathcal{A} \longrightarrow \mathcal{B}$$

which is

1. an algebra homomorphism, $\varphi \circ m = m' \circ \varphi$
2. a coalgebra homomorphism, $(\varphi \otimes \varphi) \circ \Delta = \Delta' \circ \varphi$
3. a Hopf algebra homomorphism, that is $\varphi \circ S = S' \circ \varphi$.

Moreover it has to be compatible with the action on the algebra of fields:

$$\varphi\widetilde{(\alpha \triangleright_\star \phi)}(p) = \varphi\widetilde{(\alpha)} \triangleright \varphi\widetilde{(\phi)}$$

In our case the map is an algebra homomorphism, as we said, but is not a coalgebra homorphism, since the coproduct, counit and antipode are not mapped one into the other