Noncommutative Geometry

Between Quantum Mechanics, Quantum Gravity

and the Standard Model

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In this talk I will give a non-technical, non-rigorous, overview of the research I have been doing for the past years. Mostly in collaboration with Patrizia Vitale and now Max Kurkov and Agostino Devastato, but there have been collaborations with other past and presents members of the department in basically all of the aspects i will describe, and some of the aspect I will not touch upon. I will not give detail, preferring general descriptions.

I will discuss the following points:

1. The need for a noncommutative geometry
2. The algebraic description of ordinary geometry and the possibility to generalize it
3. The noncommutative geometry of the quantum phase space
4. Field theory on noncommutative spaces
5. Noncommutative geometry and the standard model coupled with gravity

The need for a noncommutative geometry

Classical mechanics can be seen as the study of the geometry of phase space (or position momentum space). Given an initial position of a system, the classical dynamics describes its evolution.

We can have the case of constrained mechanics, the infinite dimensional case, and also the relativistic case

Relativity is a big change, from space we go to spacetime, but we still have points (events).

Even with general relativity, and the curvature of spacetime, the underlying space is still has a classical geometry.

All this changes dramatically with quantum mechanics

A quick way to see that in quantum mechanics the concept of point (of phase space) is not valid is given by the Heisenberg Microscope

The idea is that to "see" something small, of size of the order of Δx, we have to send a "small" photon, that is a photon with a small wavelength λ, but a small wavelength means a large momentum $p = h/\lambda$. In the collision there will a transfer of momentum, so that we can capture the photon. The amount of momentum transferred is uncertain.

In quantum mechanics a point in phase space is an untenable concept because of the Heisenberg uncertainty principle:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

We know what has happened. The observables, which in classical mechanics are commutative functions on the phase space, have become noncommutative operators on the Hilbert space of wave functions

A (pure) state in classical mechanics is a point of phase space, an observable is something which gives a real number for each state (the value of the function on the point)

A (pure) state in quantum mechanics is a vector on the Hilbert space, an observable is something which gives a real number for each state (the expectation value of the operator)

The difference is noncommutativity

There is an important mathematical result: The information on the (classical) phase space is encoded in the (commutative) algebra of observables, i.e. in the functions on the space

A theorem (Gelfand-Naimark) shows a complete equivalence between commutative C^*-algebras and Hausdorff topological spaces

A Hausdorff space is one for which points are separable. A C^*-algebra is an associative algebra with a norm and a complex conjugation

Given an Hausdorff space it is always possible to construct a commutative C^*-algebra: continuous complex valued functions.

The converse is also true, an arbitrary C^*-algebra is always the algebra of continuous complex valued functions on some Hausdorff space. The points of the space are the pure states of the algebra, the topology is given by convergence.

This duality has led some people, starting with Von Neumann, but principally Alain Connes, to an attempt to transcribe all properties of ordinary spaces in algebraic terms

Thus the emphasis in the description of geometry switches from points to fields

The topology is encoded by the algebra, which can always be represented as operators on some Hilbert space (loosely speaking, every algebra is a matrix algebra, possibly infinite dimensional)

The metric structure is encoded in a (generalized) Dirac operator D, which “knows” about the metric, and is used to build the differential calculus (forms). Integrals become traces of operators with the inverse of D playing the role of the measure

What if the algebra is noncommutative?

Noncommutative Spaces

If the algebra is noncommutative the identification of points with pure states (or irreducible representations) fails. Often the Hausdorff topology gives a single points

Nevertheless the topological information about the space is encoded in the noncommutative algebra. This algebra can always be represented as operators on an Hilbert space, and further geometrical properties, such as the metric, can be encoded in the generalized Dirac operator $\boxed{D}$ operator

If we succeed in transcribing objects of ordinary geometry in algebraic terms, then the generalization is "simply" done just assuming that the algebra is noncommutative

Quantum phase space is a noncommutative space, but what are its relations with the classical space?, With its structures?

Deformation of spaces

Take the algebra of classical observables, functions multiplied with the commutative product, and introduce a deformed (Gronewöld-Moyal) product:

$$(f \star g)(x,p) = f e^{\frac{i\hbar}{2}\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x}} g = fg + \frac{i\hbar}{2}(\partial_x f \partial_p g - \partial_p f \partial_x g) + O(\hbar^2)$$

So that to first order in $\hbar$

$$f \star g - g \star f = i\hbar\{f,g\} + O(\hbar^2)$$

This is a concrete realization of Dirac's correspondence principle.

The commutator is a deformation of the Poisson bracket, in the limit $\hbar \to 0$ one finds again the classical structure

This is a way to describe quantum mechanics as a deformation of classical mechanics

The usual phase is still there, but the functions defined on it, the observables form now a noncommutative algebra

It is possible to consider different deformed products, for example one which reproduces normal ordered products of operators. They correspond to different quantization schemes

The noncommutative structure of spacetime

So far we have been discussing the noncommutativity of phase space. In quantum mechanics however configuration space is still an ordinary space

Is it legitimate to expect the usual geometry to hold to all scales?

There are several arguments which indicate physical reasons for which it should not be so

Just to mention one (Bronstein), a variation of the Heisenberg microscope, at the same caricature level I used before:

In order to "measure" the position of an object, and hence the "point" in space, one has use a very small probe, and quantum mechanics forces us to have it very energetic, but on the other side general relativity tells us that if too much energy is concentrated in a region a black hole is formed.

The scale at which this happens is of the order of Planck's length

$$\ell_P = \sqrt{\frac{G\hbar}{c^3}} = 1.6 \ 10^{-33} \ cm.$$

This is the region in which the theory to use is Quantum Gravity. Unfortunately a theory we do not yet have

In fact the two problems are related. A quantum gravity theory needs spacetime to be a different object from the one used in classical geometry

For example in loop quantum gravity 3-space is directly quantized and the geometry used there (spin networks) is certainly different from the classical one

Also in string theory spacetime undergoes changes

It is not anymore a given starting point but for example its dimensions emerge from the quantization of a conformal two-dimensional field theory

Interacting strings are described by the insertion of vertex operators on the worldsheet

At ultra high energy the structure of spacetime is again a (still somewhat mysterious) object in which ordinary spacetime has undergone strong transformations (M theory)

Although people were doing noncommutative geometry applied to physics from the early nineties (more on this later), impulse to study noncommutative spaces in physics came undoubetly from Strings Frohlich-Gawedski, Landi-FL-Szabo, Seiberg-Witten when it turned out that, in some limit, the vertex operators of a string theory show the behaviour given by noncommutative coordinates

In the spirit of what I said before one can threat a noncommuting space deforming the algebra of functions with a $\star$ product similar to the one introduced in quantum mechanics, with $\hbar$ replaced by an antisymmetric matrix θ:

$$f \star g = f e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}} g$$

In this way we encode the noncommutativity of spacetime in the deformation of the algebra

Noncommutative Field Theory

Deform of a commutative theory with the presence of a star product among the fields. For example

$$S = \int d^d x \partial_\mu \varphi \star \partial^\mu \varphi + m^2 \varphi \star \varphi + \frac{g^2}{4!} \varphi \star \varphi \star \varphi \star \varphi$$

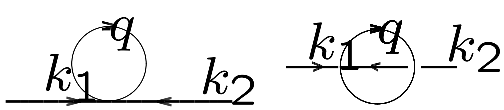
For the Grönewold-Moyal product the $\star$ on the first two terms is redundant because $\int d^d x f \star g = \int d^d x f g$

What physics comes out of these theories?

The free theory is unchanged because of the integral property. But the vertex gets a phase. For the example $\varphi^{\star 4}$:

$$V = (2\pi)^4 g \delta^4 \left(\sum_{a=1}^{4} k_a \right) \prod_{a<b} \mathrm{e}^{-\frac{i}{2}\theta^{\mu\nu} k_{a\mu} k_{b\nu}}$$

The vertex is not anymore invariant for exchange of the momenta (only for cyclic permutations), and causes a difference between planar and nonplanar diagrams



A consequence of this is Ultraviolet/Infrared Mixing Minwalla-Seiberg-Van Raamnsdong. The phenomenon for which some ultraviolet divergences disappear, just to reappear as infrared divergences

If we take seriously the fact that the world is described by this kind of non-commutative field theory which are the consequences? How do we measure $\boxed{\theta^{\mu\nu}}$, a quantity of the order of $\boxed{\ell_P^2}$?

At this level $\boxed{\theta^{\mu\nu}}$ is a background quantity, which selects two directions in space (analog of electric and magnetic fields). Their presence breaks Lorentz invariance and the noncommutativity will have left its imprinting in the early universe

Direct accelerator measurements are more difficult because the earth rotation washes up the effect. But one can look for otherwise forbidden processes

It is not easy however to distinguish predictions coming from these kind of theories from other breakings of Lorentz invariance

deformations of spacetime may require a deformation of symmetries: quantum groups. This will take another seminar!

Connes' approach to the standard model and the Higgs

While the formalism is geared towards the construction of genuine noncommutative spaces, spectacular results are obtained considering almost commutative geometries, which leads to: **Connes' approach to the standard model**

The project is to transcribe electrodynamics on an ordinary manifold using algebraic concepts: The algebra of functions, the Dirac operator, the Hilbert space and chirality and charge conjugation. One can then write the action in purely algebraic terms.

Then the machinery can be applied to noncommutative space, or in general to other algebras.

In this case the space is only "almost" commutative, in the sense that there still is an underlying spacetime, the noncommutative algebra describing space is said to be Morita equivalent to a commutative algebra

For the full standard the algebra is a tensor product $\mathcal{A} = C(\mathbf{R}^4) \otimes \mathcal{A}_F$, with $\mathcal{A}_F$ a finite matrix algebra of 3×3 matrices, quaternions (which are matrices of the kind $a^\mu \sigma_\mu$) and complex numbers corresponding to $SU(3), SU(2)$ and $U(1)$ respectively.

The information about masses and Cabibbo mixing are encoded in the D operator

There is a translation in algebraic terms of the requirement that a generic topological space is a manifold (i.e. it has a differential structure). This is a set of seven purely algebraic conditions on the algebra, the Hilbert space and the D_0, γ and J operators.

Application of these conditions to the almost commutative geometry, plus the imposition of chirality, select the gauge group to be $SU(3) \times SU(2) \times U(1)$.

The model, especially in its last version (Chamseddine-Connes-Marcolli) has some predictive power (mass of the Higgs). More on this later.

The presence of chirality $\gamma = \gamma^\dagger$, with $\gamma^2 = 1$, the generalization of γ_5, causes the splitting

$$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R$$

Eigenspaces of $\frac{1}{2}(1 \pm \gamma)$

There is other operator, J, charge conjugation. It has important mathematical connections, Tomita-Takesaki operator, KMS states, but i will not discuss much it in this talk

The central idea behind spectral geometry is that these ingredients are sufficient to describe not only a geometry, but also the behaviour of the fields defined on them, and their couplings to the geometry of spacetime (gravity). Treating on an equal footing the *external geometry (spacetime), with the inner one, gauge degrees of freedom*

The main success of this view is the spectral action. The algebra is the product of the algebra of functions on spacetime, the Hilbert space is that of fermion matter fields, and the Dirac operator contains all information on the metric of spacetime, as well as the masses , couplings and mixings of fermions.

The spectral action contains two part, one is the bosonic action, to be read in a Wilsonian renormalization group sense:

$$S_B = \mathrm{Tr}\,\chi\left(\frac{D_A}{\Lambda}\right)$$

where $D_A = D_0 + A$ is a fluctuation of the Dirac operator, χ is the characteristic function of the interval $[0,1]$, or some smoothened version of it, and Λ is a cutoff

Then there is a "standard" fermionic action $\langle\Psi|\, D_A\, |\Psi\rangle$

The bosonic action is finite by construction, the fermionic part needs to be regularized

In the work of Chamseddine, Connes and Marcolli the renormalization group flow is done by considering as boundary condition the unification of the three interaction coupling constants at Λ. This is approximately (but not exactly) true.

The various couplings and parameters are then found at low energy via the renormalization flow

Yukawa couplings (masses) and mixings are taken as inputs. The mass parameter of the Higgs is however not needed, and is a function of the other parameters (which are dominated by the top mass).

There is therefore predictive power, and the mass of the Higgs is found to be $\sim 170\text{GeV}$. A value experimentally disfavoured. Nevertheless It is fascinating that a theory without so little input finds a Higgs mass relatively close to the expected (and possibly measured) value

Technically the bosonic spectral action is a sum of residues and can be expanded in a power series in terms of Λ^{-1} as

$$S_B = \sum_n f_n \, a_n(D^2/\Lambda^2)$$

where the f_n are the momenta of χ

$$\begin{aligned} f_0 &= \int_0^\infty \mathrm{d}x \, x\chi(x) \\ f_2 &= \int_0^\infty \mathrm{d}x \, \chi(x) \\ f_{2n+4} &= (-1)^n \partial_x^n \chi(x)\Big|_{x=0} \quad n \geq 0 \end{aligned}$$

the a_n are the Seeley-de Witt coefficients which vanish for n odd. For D^2 of the form

$$D^2 = -(g^{\mu\nu}\partial_\mu\partial_\nu \mathbb{I} + \alpha^\mu \partial_\mu + \beta)$$

defining (in term of a generalized spin connection containing also the gauge fields)

$$\begin{aligned}\omega_\mu &= \frac{1}{2}g_{\mu\nu}\left(\alpha^\nu + g^{\sigma\rho}\Gamma^\nu_{\sigma\rho}\mathbb{I}\right)\\ \Omega_{\mu\nu} &= \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu,\omega_\nu]\\ E &= \beta - g^{\mu\nu}\left(\partial_\mu\omega_\nu + \omega_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho\right)\end{aligned}$$

then

$$\begin{aligned}a_0 &= \frac{\Lambda^4}{16\pi^2}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,\mathbb{I}_F\\ a_2 &= \frac{\Lambda^2}{16\pi^2}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\left(-\frac{R}{6}+E\right)\\ a_4 &= \frac{1}{16\pi^2}\frac{1}{360}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,(-12\nabla^\mu\nabla_\mu R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu}\\ &\quad +2R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 60RE + 180E^2 + 60\nabla^\mu\nabla_\mu E + 30\Omega_{\mu\nu}\Omega^{\mu\nu})\end{aligned}$$

tr is the trace over the inner indices of the finite algebra $\mathcal{A}_F$ and in Ω and E are contained the gauge degrees of freedom including the gauge stress energy tensors and the Higgs, which is given by the inner fluctuations of D

There is however an intimate connection between the fermionic and the bosonic action (AAA,FL). Consider the fermionic action alone, a theory in which fermions move in a fixed background

The classical action is invariant for the following transformation

$$|\Psi\rangle \to \mathrm{e}^{\frac{1}{2}\phi} |\Psi\rangle$$

$$D \to \mathrm{e}^{-\frac{1}{2}\phi} D \mathrm{e}^{-\frac{1}{2}\phi}$$

Recalling the presence of $\sqrt{\det g}$ in the integral for the position representation of the Hilbert space it is easy to see that this is actually related to Weyl rescaling

$$g^{\mu\nu} \to \mathrm{e}^{2\phi} g^{\mu\nu}$$

This is a however symmetry of the classical action, not of the regularized quantum partition function: anomaly:

$$Z(D) = \int [\mathrm{d}\psi][\mathrm{d}\bar{\psi}] e^{-S_\psi}$$

We will consider a theory in which the symmetry is explicitly broken by a physical scale and look for the anomalous and not anomalous parts of the partition function

Introducing a normalization dimensional constant μ the partition function can be formally written as a determinant, :

$$Z(D,\mu) = \int [\mathrm{d}\psi][\mathrm{d}\bar{\psi}] e^{-S_\psi} = \det\left(\frac{D}{\mu}\right)$$

The determinant is still infinite and we need to introduce a cutoff

In the spirit of noncommutative geometry the most natural way to regularize is a truncation of the spectrum of the Dirac operator. This was considered long ago by Andrianov, Bonora, Fujikawa, Novozhilov, Vassilevich

The cutoff is enforced considering only the first N eigenvalues of D

Consider the projector $P_N = \sum_{n=0}^{N} |\lambda_n\rangle \langle\lambda_n|$ with λ_n and $|\lambda_n\rangle$ the eigenvalues and eigenvectors of D

N is a function of the cutoff defined as $N = \max n \text{ such that } \lambda_n \leq \Lambda$

We effectively use the N^{th} eigenvalue as cutoff

The choice of a sharp cutoff could be changed in favour of a cutoff function, similar to the choice of χ

The bosonic action is then induced by the fermionic one by the renormalization flow AAA, FL.

The calculation, based on the transformation properties of the partition function, is standard and I omit it

Consider the Dirac operator for the standard model, in its barest essentiality (for our purposes). In the left-right splitting of $\mathcal{H}$, the operator D it is a 2×2 matrix

$$D = \begin{pmatrix} i\gamma^\mu D_\mu + \mathbb{A}_L & \gamma_5 S \\ \gamma_5 S^\dagger & i\gamma^\mu D_\mu + \mathbb{A}_R \end{pmatrix}$$

where

$D_\mu = \partial_\mu + \omega_\mu$, ω_μ the spin connection. $\mathbb{A}$ contains all gauge fields, S contains the Higgs field H, Yukawa couplings, mixings. . .

The calculation is performed splitting the partition function in the product of a term invariant for Weyl transformations, and another not invariant, which will depend on the field ϕ, the dilaton.

Looking at the role of the dilaton in the partition function it iis possible to see that it is a collective mode of fermions, and is mediating the breaking of the symmetry

We assume therefore the presence, in an earlier epoch, of a conformal point, in which the symmetry is restored. A phase in which all particles are massless, and the Higgs potential does not have the degenerate minimum

The calculation, done so far neglecting variations of the field, of the bosonic action gives a slight modification of the spectral action

The behaviour of D under Weyl rescaling gives the transformation of H under such transformation. Only the H^4 term in the effective potential is invariant, and it can be multiplied by a constant quantity (ϕ_0). This gives, in this approximation, the invariant part of the effective potential

The other terms of the effective potential can be calculated using the heat kernel. We finally obtain an effective potential

$$V = V_0 + Ae^{4\phi} + BH^2e^{2\phi} - CH^4 + EH^2$$

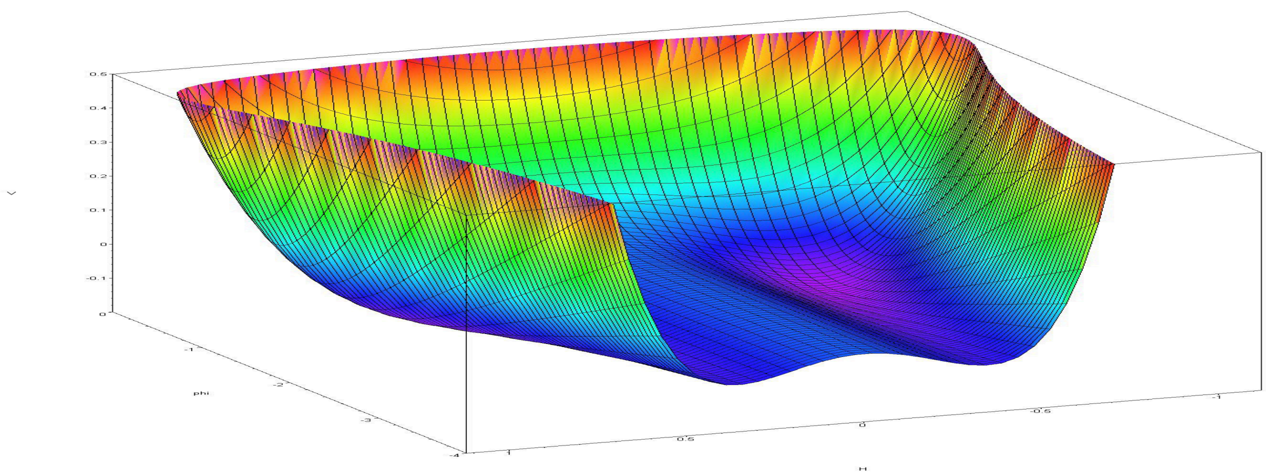
The coefficients are functions of the the parameters Λ and μ and another integration constant.

The normalization constant μ can be fixed requiring that the constant term in the action proportional to Λ^4 vanishes.

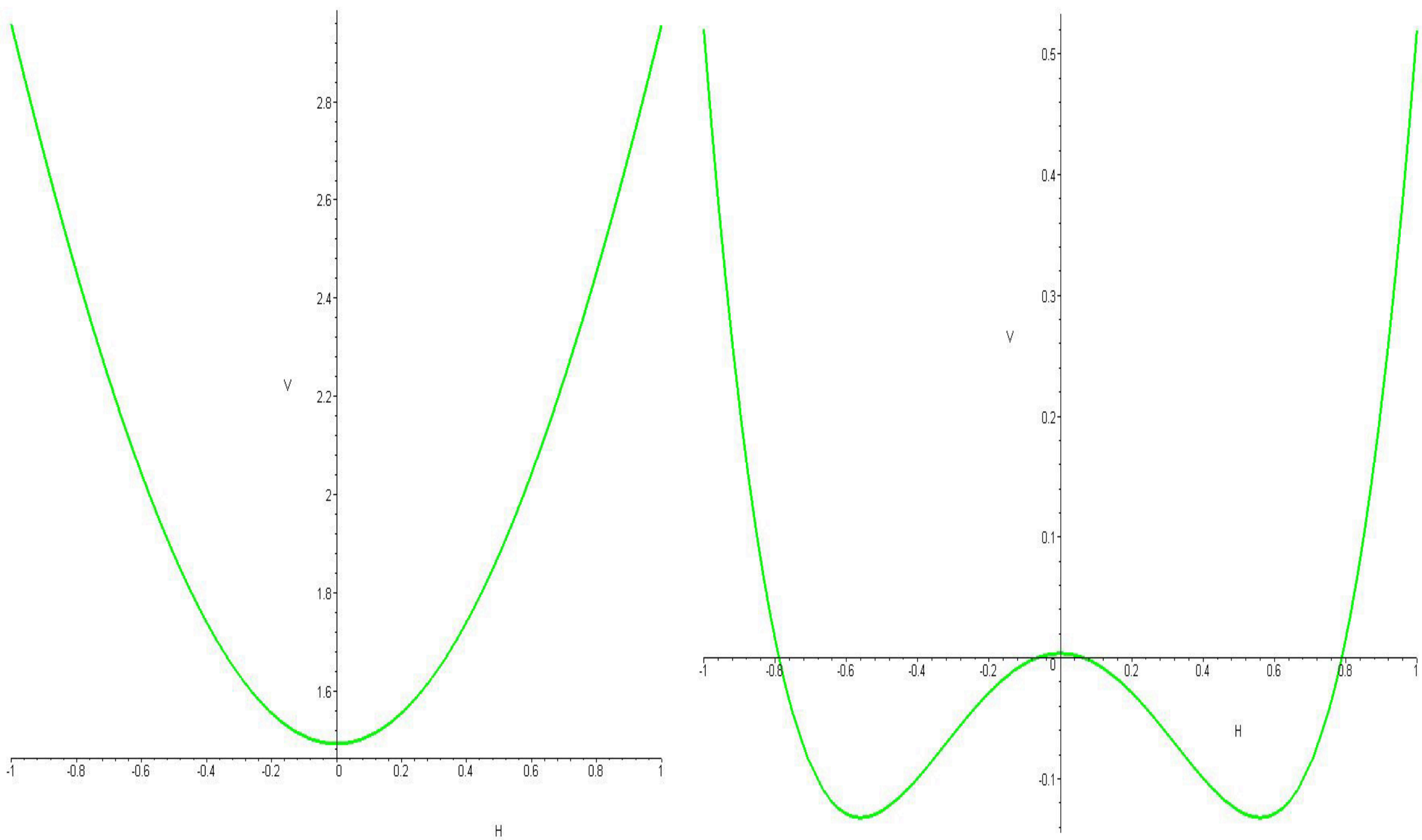
Properties of the potential:

- The existence of a local minimum
- The existence of an unbroken phase from which the potential may roll down to the broken phase

Plot of the effective Higgs-Dilaton potential:



We see that for different values of ϕ, the potential $V(H)$ has a transition from a symmetric to a broken phase.



What I have describe so far referred to the case of a constant dilaton.

It is possible to actually calculate explicitly the action of the collective modes:

$$S_{coll} = \Lambda^2 \left(\alpha_1 (H^2 - \alpha_2 R)(e^{2\phi(x)} - 1) + \alpha_3 e^{2\phi(x)} \phi_{;\mu}\phi_{;}{}^{\mu}\right)$$

$$+ \alpha_4 \left(\phi_{;\mu}{}^{\mu} + \phi_{;\mu}\phi_{;}{}^{\mu}\right)\left(\alpha_5 H^2 - \frac{1}{3}\cdot R - \phi_{;\mu}{}^{\mu} - \phi_{;\mu}\phi_{;}{}^{\mu}\right)\Big)$$

$$-\alpha_6 \phi(x) \left(H_{;\mu}H_{;}^{\mu} - \frac{1}{6} R H^2 + \alpha_7 H^4 + \alpha_8 C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}\right.$$

$$\left. + \alpha_9 F^{\mu\nu}F_{\mu\nu} + \alpha_{10} W_{\mu\nu}W^{\mu\nu} + \alpha_{11} G_{\mu\nu}G^{\mu\nu}\right) + \text{invariant part}$$

where the α_i are positive numbers, which depend on the couplings

The qualitative behaviour does not change, we are in the process of “putting the numbers in”

Things I did not even mention, and on which I worked:...

- Quantum symmetries of noncommutative field theories
- The noncommutative geometry of string theories (Vertex operators, Hilbert space of string states, duality etc.
- Fuzzy spheres and discs, field theories on fuzzy spaces
- Noncommutative (non Hausdorff) lattices
- Noncommutative tori and other noncommutative spaces
- Matrix models

And then there are the things i did not mention and not worked on...

No conclusions!

We are far form having finished working on this subject

There are lots of thins to do

Anyone wants to help?