# Noncommutative Geometry

&

# the Standard Model

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Prague 2010

Let me start with a definition of Geometry

For which I can think of no higher authority than **Wikipedia**

*Geometry (Greek $\gamma\varepsilon\omega\mu\varepsilon\tau\rho\iota\alpha$; geo = earth, metria = measure) is a part of mathematics concerned with questions of size, shape, and relative position of figures and with properties of space.*

Geometry is at the hearth of several physical theories, including classical mechanics, special and general relativity, strings . . .

In all these theories the geometry used is the mostly usual one, based on the concepts of points, lines etc.

**Is it legitimate to expect the usual geometry to hold to all scales?**

We all know that already at the level of quantum mechanics we have to abandon the classical geometry of phase space. There are several arguments which indicate physical reasons for which also the geometry of spacetime must be a quantum one

Just to mention one (Bronstein, Doplicher-Fredenhagen-Roberts) which evocates a reasoning similar to the heisenberg microscope for phase space:

In order to "measure" the position of an object, and hence the "point" in space, one has use a very small probe, and quantum mechanics forces us to have it very energetic, but on the other side general relativity tells us that if too much energy is concentrated in a region a black hole is formed.

The scale at which this happens is of the order of Planck's length

$$\ell_P = \sqrt{\frac{G\hbar}{c^3}} = 1.6\ 10^{-33}\ cm.$$

This is the region in which the theory to use is Quantum Gravity. Unfortunately a theory we do not yet have

In fact the two problems are related. A quantum gravity theory needs spacetime to be a different object from the one used in classical geometry

The natural arena for a noncommutative geometry is quantum gravity, nevertheless the structure of spacetime can have consequences also in the symmetry structures which constitute the standard model of elementary particles

In fact the first appearances of noncommutative geometry in physics have been in the attempts to explain the properties of the standard model and the Higgs mechanism.

Spacetime in this case is described by an "almost commutative geometry", Madore, Dubois-Violette-Kerner-Madore Connes-Lott, . . .

In these models spacetime is the usual one, but there is an "internal" noncommutative structure which carries information on the nontrivial symmetries of the model

Then there is a "natural" action which reproduces the characteristics of the model, possibly including gravity as well

In Connes' view the physics of the standard model plus gravity is inserted in more general programme aimed at translating all concepts of ordinary geometry in an algebraic framework, which opens the possibility to generalize all concepts in the noncommutative framework.

Three main ingredients form the Spectral Triple (plus some seasonings)

- A $C^*$-algebra $\mathcal{A}$ encodes the topology of spacetime
- A Hilbert space $\mathcal{H}$ on which the algebra is represented as bounded operators, and which gives the matter content of the theory
- A Generalization of the Dirac operator $D$ which gives the differential and metric structures, and whose fluctuations give the action
- The seasoning are the chiral structure $\gamma$, and the real structure $J$ given by the generalization of the charge conjugation operator

Connes' approach the standard model is aimed at understanding the geometry of it. To use his words one has to "twist" the geometry to make it fit the standard model and gravity.

The game is then to see which sets of data (an algebra, a Hilbert space, a Dirac operator) reproduce the standard model

The algebra is the product of the algebra of functions on space-time times a finite dimensional matrix algebra

$$\mathcal{A} = C(\mathbb{R}^4) \otimes \mathcal{A}_F$$

Likewise the Hilbert space is the product of fermions times a finite dimensional space which contains all matter degrees of freedom, and also the Dirac operator contains a continuous part and a discrete one

$$\mathcal{H} = \mathsf{Sp}(\mathbb{R}^4) \otimes \mathcal{H}_F$$

$$D = \gamma^\mu \partial_\mu \otimes \mathbb{I} + \gamma \otimes D_F$$

In its most recent form (Chamseddine-Connes-Marcolli) a crucial role is played by the mathematical requirements that the non-commutative algebra satisfies the requirements to be the non-commutative generalization of a manifold

Then the internal algebra, is almost uniquely derived to be

$$\mathcal{A}_F = \mathbf{C} \oplus \mathbb{H} \oplus M_3(\mathbf{C})$$

For example if the spacetime is two copies of a manifold the generalization of the electrodynamics action gives a $U(1) \times U(1) \to U(1)$ Higgs mechanism

Instead for an algebra given by functions on spacetime with values in $\mathbb{C} \otimes \mathbb{H} \otimes M_3$, (complex numbers, quaternions, three by three matrices) we obtain the standard model

The finite part of the Dirac operator $D_F$ contains all informations about fermion masses and coupling

Central to this construction is the action, purely based on spectral properties of a covariant Dirac operator

Consider the covariant operator

$$D_A = D + A$$

Where $A$ is the connection which naturally comprises all the fluctuations of the "metric". The internal part of the algebra gives the inner gauge group, while the fluctuations of the continuous part give the Levi-Civita conection

The action is

$$S = S_B + S_F = \mathrm{Tr}\,\chi\left(\frac{D_A^2}{\Lambda^2}\right) + \langle\Psi|\, D_A \,|\Psi\rangle$$

With $\Lambda$ a cutoff in Wilsonian sense, and $\chi$ some possibly smoothened version of the step function

The bosonic action, in the case of $\chi$ the step function, is just the number of eigenvalues smaller than the cutoff

It can be evaluated using heath kernel techniques and the final result gives the action of the standard model coupled with gravity.

The fascinating aspect of this theory is that the Higgs appears naturally as the “vector” boson of the internal noncommutative degrees of freedom. In the process of writing the action all masses and coupling are used as inputs, but one saves one parameter.

The Higgs mass is predicted, in the present form of the model, to be $\sim 170\text{GeV}$. A value too small and experimentally disfavoured.

Technically the bosonic spectral action is a sum of residues and can be expanded in a power series in terms of $\Lambda^{-1}$ as

$$S_B = \sum_n f_n \, a_n(D^2/\Lambda^2)$$

where the $f_n$ are the momenta of $\chi$ and the $a_n$ are the Seeley-de Witt. For $D^2$ of the form

$$D^2 = g^{\mu\nu}\partial_\mu\partial_\nu \mathbb{1} + \alpha^\mu\partial_\mu + \beta$$

we have

$$\begin{aligned}
\omega_\mu &= \frac{1}{2} g_{\mu\nu}\left(\alpha^\nu + g^{\sigma\rho}\Gamma^\nu_{\sigma\rho}\mathbb{1}\right) \\
\Omega_{\mu\nu} &= \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu] \\
E &= \beta - g^{\mu\nu}\left(\partial_\mu\omega_\nu + \omega_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho\right)
\end{aligned}$$

then

$$\begin{aligned}
a_0 &= \frac{\Lambda^4}{16\pi^2}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,\mathbb{1}_F \\
a_2 &= \frac{\Lambda^2}{16\pi^2}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\left(-\frac{R}{6} + E\right) \\
a_4 &= \frac{1}{16\pi^2}\frac{1}{360}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,(-12\nabla^\mu\nabla_\mu R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} \\
&\quad +2R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 60RE + 180E^2 + 60\nabla^\mu\nabla_\mu E + 30\Omega_{\mu\nu}\Omega^{\mu\nu})
\end{aligned}$$

tr is the trace over the inner indices of the finite algebra $\mathcal{A}_F$ and in $\Omega$ and $E$ are contained the gauge degrees of freedom including the gauge stress energy tensors and the Higgs, which is given by the inner fluctuations of $D$

There are other problems with this approach, it is Euclidean, fine tuning is needed, the coupling constants all meet in one point the quantization is done in the "commutative" way, which is somehow anticlimactic.

The fact that the model is "ad hoc" and in the end it writes a known action is not a problem. The programme was to fit the standard model into a more general framework, not to derive it form an higher theory

Once the framework is known one can try to understand where it comes from.

The model is probably not yet ready to give trustful experimental predictions, but it is important to constantly update these prediction to understand in which direction the refinements are needed.

The reason to be of the construction is that it is made using the spectral properties of the noncommutative geometry, and as such is it immediately ready for noncommutative generalizations or deformations of spacetime

Wilson's renormalization becomes the fact that the cutoff is just the truncations of the higher eigenvalues of $D_A$, that is the ultraviolet components of the geometry. Since the standard model may be an effective theory, the cutoff may have a physical meaning of the limit of validity of this almost commutative geometry, leading to a fully noncommutative one.

The matter content, and the fermionic action, is however treated in the standard way

It is possible to show however that bosonic part of the spectral action can be obtained as the contribution to the action necessary to cancel the scale anomaly

In the usual treatment of the spectral action the bosonic part is, given $\Lambda$ already finite, while the fermionic action must be regularised, and this is done using standard techniques

Start with just a theory in which some fermions are coupled to some background, this background is fixed because we have not considered the bosonic part of the action.

I may take the background to be flat, but this is not necessary. Note the similarities with Sakharov emergent gravity at one loop, and Steinacker emergent gravity from matrix models to be discussed later

The theory has a (global for the time being) classical symmetry for scale invariance

$$x^\mu \to \mathrm{e}^{\phi} x^\mu \ , \psi \to \mathrm{e}^{-\frac{3}{2}\phi}\psi \ , D \to \mathrm{e}^{-\frac{1}{2}\phi} D \mathrm{e}^{-\frac{1}{2}\phi}$$

and the action is formally a determinant, which needs regularization

$$Z(D) = \int [\mathrm{d}\psi][\mathrm{d}\bar{\psi}] e^{-S_\psi} = \det(D)$$

The term one has to add in then basically the bosonic spectral action (with a sharp cutoff), with some minor changes in the Seely-De Witt coefficients (FL, Andrianov)

In some sense God create matter before light!

A parallel development, initiated by Doplicher-Fredhenagen-Roberts is to mimic for spacetime what happens for the quantum phase space. loosely speaking one has *noncommuting variables*

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$

$\theta$ is a central operator in the original construction, but often considered a constant in other developments, also coming from string (Seiberg-Witten)

In the spirit of what I said before one can threat a noncommuting space deforming the algebra of functions with a Grönewold-Moyal $\star$ product:

$$f \star g = f e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}} g$$

In this way we encode the noncommutativity of spacetime in the deformation of the algebra. These theories are by far the most studied noncommutative geometries, including some of their phenomenological and cosmological consequences of them

I will to discuss them and stay at a more abstract level: Matrix models. in = the work of Steinacker and others (Aschieri, Chatzistavrakidis, Grammatikopoulos, Grosse, Klammer, FL, Wohlgenannt, Zoupanos . . . )

The rationale behind this approach is quite simple and can be very heuristically (and therefore incorrectly) be stated as follows

- A noncommutative geometry is described by a noncommutative algebra, deformation of the commutative algebra of function on some space

- Any noncommutative ( $C^*$ )-algebra is represented as operators on some Hilbert space

- Operators on an Hilbert space are just infinite matrices

The objects we have defined are elements of a noncommutative algebra and we can always represent them as operators on a Hilbert space, in this case the integral becomes a trace and this suggests the use of the matrix action

$$S = -\frac{1}{4g}\,\mathrm{Tr}\,[X^\mu, X^\nu][X^{\mu'}, X^{\nu'}]g_{\mu\mu'}g_{\nu\nu'}$$

Where the $X$'s are operators (matrices) and the metric $g_{\mu\mu'}$ is the flat Minkowski (or Euclidean) metric

The fascinating characteristic of this action is that gravity emerges from it.

The equations of motion are

$$[X^\mu, [X^\nu, X^{\mu'}]] g_{\mu\mu'} = 0$$

A possible vacuum (the $U(1)$ Moyal vacuum) given by a set of matrices $X_0$ such that $[X_0^\mu, X_0^\nu] = \mathrm{i}\,\theta^{\mu\nu}$ con $\theta$ constant

This is some sort of semiclassical vacuum and we can consider $f(X_0)$ as deformation of functions on a Moyal space

Now let fluctuate these “coordinates” and consider $X^\mu = X_0^\mu + A^\mu$ so that $[X^\mu, X^\nu] = \mathrm{i}\,\theta(X)$ and we are considering a nonconstant noncommutativity

Gravity emerges as nontrivial curvature considering for example the coupling with a scalar field $\Sigma$. The (free action) is

$$\mathrm{Tr}\,[X^\mu, \Sigma][X^\nu, \Sigma]g_{\mu\nu} \sim \int \mathrm{d}x (D_{\mu'}\Sigma)(D_{\nu'}\Sigma)\theta^{\mu\mu'}\theta^{\nu\nu'}g_{\mu\nu} = \int \mathrm{d}x (D_\mu\Sigma)(D_\nu\Sigma)G^{\mu\nu}$$

where we have defined the new, (non flat) metric $G^{\mu\nu}(x) = \theta^{\mu\mu'}\theta^{\nu\nu'}g_{\mu'\nu'}$

Effectively a curved background has emerged from noncommutativity. The gravitational action is recovered as effective action at one loop

An alternative vacuum, still solution of the equations of motion, is

$$\bar{X}_0^\mu = X_0^\mu \otimes \mathbb{1}_n$$

Consider as the fluctuations

$$X^\mu = \bar{X}_0^\mu = \bar{X}_0^\mu + A_0^\mu + A_\alpha^\mu \lambda_\alpha$$

where in the fluctuations we have separated the traceless generators of $SU(n)$ from the trace part ( $A_0$ )

The $U(1)$ trace part of the fluctuation gives rise to the gravitational coupling, while the remaining $A_\alpha$ describe a $SU(n)$ gauge theory

We are slightly better than usual noncommutative geometry models which have $U(n)$ symmetry. How to get closer to the standard model?

The plan is to find a matrix model with some coordinates (a vacuum) and an action which reproduces, as close as possible, the standard model

We should not be shy of making as many assumptions as are needed. The game is not to find the standard model, but rather to find a noncommutative geometry which "fits" it

We have already managed to find a $SU(n)$ theory. We need two more stages, first a modification of the model to allow $SU(3) \times SU(2) \times U(1)$, and then a symmetry breaking mechanism

We also have to put fermions (in the right representation)

The first stage can be accomplished by considering the following vacuum with an extra coordinate, for which I will use the index $\boldsymbol{\Phi}$ and a different typeset to differentiate it from the usual coordinates

$$\mathfrak{X}^{\Phi} = \begin{pmatrix} \alpha_1 \mathbb{1}_2 & & \\ & \alpha_2 \mathbb{1}_2 & \\ & & \alpha_3 \mathbb{1}_3 \end{pmatrix}$$

with $\alpha_i \in \mathbf{R}$

Since $[X^{\mu}, \mathfrak{X}^{\Phi}] = 0$ the equations of motion are still satisfied, but the gauge symmetry is reduced to

$$SU(2) \times SU(2) \times SU(3) \times U(1) \times U(1)$$

The contribution to the action is a term of the kind

$$F^{\mu\Phi} = [\bar{X}^\mu + A^\mu, A^\Phi] + [A^\mu, X^\Phi]$$

with

$$[\bar{X}^\mu + A^\mu, \mathfrak{X}^\Phi] = i\theta^{\mu\nu} D_\nu \mathfrak{X}^\phi = i\theta^{\mu\nu}(\partial_\nu + iA_\nu)\mathfrak{X}^\phi, =$$

$$-(2\pi)^2 \,\mathrm{Tr}\, [X^\mu, \mathfrak{X}^\phi][X^\nu, \mathfrak{X}^\phi]\eta_{\mu\nu} = \int d^4x G^{\mu\nu} \left(\partial_\mu \mathfrak{X}^\Phi \partial_\nu \mathfrak{X}^\Phi - [A_\mu, \mathfrak{X}^\Phi][A_\nu, \mathfrak{X}^\Phi]\right)$$

The mixed terms vanish assuming the Lorentz gauge $\partial^\mu A_\mu = 0$. Since $\mathfrak{X}^\Phi = \text{const}$ the first term in the above integral vanish

We can separate the fluctuations of this extra dimension which are a field, the (high energy) Higgs field.

Consider the block form of $A^\mu$

$$A^\mu = \begin{pmatrix} A^\mu_{11} & A^\mu_{12} & A^\mu_{13} \\ A^\mu_{21} & A^\mu_{22} & A^\mu_{23} \\ A^\mu_{31} & A^\mu_{32} & A^\mu_{33} \end{pmatrix}$$

The first term of the curvature is the covariant derivative

The second term instead is

$$[A^\mu, \mathfrak{X}^\phi] = \begin{pmatrix} 0 & (\alpha_2 - \alpha_1)A^\mu_{12} & (\alpha_3 - \alpha_1)A^\mu_{13} \\ (\alpha_1 - \alpha_2)A^\mu_{21} & 0 & (\alpha_3 - \alpha_2)A^\mu_{23} \\ (\alpha_1 - \alpha_3)A^\mu_{31} & (\alpha_2 - \alpha_3)A^\mu_{32} & 0 \end{pmatrix}$$

If the differences $\alpha_1 - \alpha_2$ is large, all non diagonal blocks of $A^\mu$ acquire large masses decoupling

We now need to introduce fermions. They are described by the matrix

$$\Psi = \begin{pmatrix} \mathcal{L}_{4\times 4} & \mathcal{Q} \\ \mathcal{Q}' & 0_{3\times 3} \end{pmatrix}$$

$L$ contains leptons (color-blind, $Q$ and $Q'$ contain quarks (which we assume to be in $(\bar{3})$ for convenience)

$$\mathcal{L} = \begin{pmatrix} 0_{2\times 2} & L_L \\ L'_L & \begin{matrix} 0 & e_R \\ e'_R & 0 \end{matrix} \end{pmatrix}$$

$$L_L = \begin{pmatrix} \tilde{l}_L & l_L \end{pmatrix}, \qquad l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \qquad \tilde{l}_L = \begin{pmatrix} \tilde{e}_L \\ \tilde{\nu}_L \end{pmatrix}$$

Here $l_L$ is the standard (left-handed) leptons, $e_R$ the right-handed electron, $\tilde{l}$ corresponds to additional leptons with the same quantum numbers as Higgsinos in principle allowed by the model.

The fields with a prime may or may not be new independent fields. They provide some sort of "mirror sector", and can be set to zero)

The quark matrix is

$$\mathcal{Q} = \begin{pmatrix} Q_L \\ Q_R \end{pmatrix}, \qquad Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad Q_R = \begin{pmatrix} d_R \\ u_R \end{pmatrix}$$

The correct hypercharge, electric charge and baryon number are then reproduced by the following traceless generators

$$Y = \begin{pmatrix} 0_{2\times 2} & & \\ & -\sigma_3 & \\ & & -\frac{1}{3}\mathbb{1}_{3\times 3} \end{pmatrix} - \frac{1}{7}$$

$$Q = T_3 + \frac{Y}{2} = \frac{1}{2}\begin{pmatrix} \sigma_3 & & \\ & -\sigma_3 & \\ & & -\frac{1}{3}\mathbb{1}_{3\times 3} \end{pmatrix} - \frac{2}{7}\mathbb{1}$$

$$B = \begin{pmatrix} 0 & & \\ & 0 & \\ & & -\frac{1}{3}\mathbb{1}_{3\times 3} \end{pmatrix} - \frac{1}{7}$$

which act in the adjoint

Weak and colour interactions sit in the first and last diagonal blocks

The charges of all fermions turn out to be the correct ones, which is non trivial, not every charge of the fermions can be obtained

The electroweak breaking can be accomplished by another extra coordinate

$$\mathfrak{X}^{\varphi} = \begin{pmatrix} 0_{2\times 2} & \varphi & 0_{2\times 1} & 0_{2\times 1} & 0_{2\times 1} & 0_{2\times 1} \\ \varphi^{\dagger} & 0 & 0 & 0 & 0 & 0 \\ 0_{1\times 2} & 0 & 0 & 0 & 0 & 0 \\ 0_{1\times 2} & 0 & 0 & 0 & 0 & 0 \\ 0_{1\times 2} & 0 & 0 & 0 & 0 & 0 \\ 0_{1\times 2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Where $\varphi$ is the usual 2-component Higgs with vacuum expectation value

$$\langle\varphi\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix}$$

We have a matrix model, based on a noncommutative spacetime, which reproduces gravity with an emergent mechanism and contains gauge theories

The same model, modulo some modifications (like soft terms in the action) contains a vacuum with a symmetry which resembles the correct gauge interactions

Therefore I claim we are "close" to phenomenology, and hence the full circle

But we are not there...

The gauge group is too big (also after the second breaking), we eliminate one $U(1)$ with gravity, but we still have unwanted generators

The Yukawa couplings pairs the correct left-right particles, but the couplings are all the same (before renormalization)

No generations

. . .

Hopefully a better understanding of the model will indicate the necessary modifications to make the model more predictive

# Conclusions

I discussed two models in which the noncommutative structure of spacetime gives indications on the properties and symmetries of the standard model

Neither model is in yet mature to really give sound phenomenological indications, of the kind you tell experimentalists, although Connes' model is more advanced

They share the feature that there some extra "coordinates" which participate in essential way to the noncommutativity, enlarging and "quantizing" the original idea of Nordstrom, Kaluza, Klein. Something John Madore has been advocating for years

Possibly also with new idea this sort of models can rise to the challenge of becoming a real tool for particle physics