Spectral Action from Anomalies

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What if we got it wrong, and
instead of the familiar

Fiat Lux

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Fiat Materia!

Our starting point is the creation some matter described by fermion fields $\boxed{\Psi}$ transforming under some reducible representation of a gauge group (such as $\boxed{SU(3) \times SU(2) \times U(1)}$)

The fermions belong to a Hilbert space of spinors $\boxed{\mathcal{H}}$, which I take to have a left and a right component

$$\boxed{\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R}$$

For the moment there are no boson fields to mediate the interaction

Once you have created fermions and symmetries you need to make them move. So you create a background for them, and a classical action to govern dynamics.

$$\boxed{\langle\Psi|\, D \,|\Psi\rangle}$$

where $\boxed{D}$ is an operator on $\boxed{\mathcal{H}}$ which I will call the Dirac operator

This operator is made of two parts acting on spinors $\boxed{\begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}}$:

$$\boxed{D = D_0 + A}$$

$$\boxed{D_0 = \begin{pmatrix} \gamma^\mu \partial_\mu & M \\ M^\dagger & \gamma^\mu \partial_\mu \end{pmatrix}}$$

Where $\boxed{M}$ contains all masses (and mixings) of the fermions and the $\boxed{\gamma}$'s are those relative to a possibly curved spacetime

The matrix $\boxed{A}$ is a fixed background gauge field.

I have implicitly introduced a (Euclidean) spacetime. And therefore at this stage I have the notion of the algebra $\boxed{\mathcal{A}}$ of smooth functions of this space time, which in this case is really a noncommutative algebra of matrix valued functions acting as operators on $\boxed{\mathcal{H}}$

I could have gone the other way around, defining from the start the spectral triple $\boxed{\mathcal{A}, \mathcal{H}, D}$

The algebra $\boxed{\mathcal{A}}$ contains all information about the topology and the differential structure of the space

The operator $\boxed{D}$ gives the metric structure

This is Connes programme of translation of all geometry into into algebraic terms, based on the spectral triple

In fact the model is ready for generalizations to genuinely non-commutative spaces (not matrix valued functions on an ordinary space). In this seminar I will not undertake this step.

I am deliberately vague as to the detail of the model at this stage, and I am not discussing important elements of the theory, like chirality or charge conjugation. What I will be discussing next is rather general and does not depend crucially on these elements

So far I have a classical theory of matter fields moving in a fixed background

The objects involved in the writing of the action have physical dimension, for example an unit of length $\boxed{\ell}$, so that I can measure volumes as $\boxed{\ell^{-4}}$, masses and the Dirac operator in general as $\boxed{\ell^{-1}}$ etc. I take the speed of light to be 1

The classical action is invariant under a change of this scale, which can also be local, (Weyl original gauge theory). Dynamics is invariant under such change.

The symmetry is:

$$\boxed{x^\mu \to \mathrm{e}^{\phi} x^\mu \ , \psi \to \mathrm{e}^{-\frac{3}{2}\phi}\psi \ , D \to \mathrm{e}^{-\frac{1}{2}\phi} D \mathrm{e}^{-\frac{1}{2}\phi}}$$

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But if the classical theory is invariant, the measure in the quantum path integral is not. We have an anomaly!

We start from the partition function

$$\boxed{Z(D) = \int [\mathrm{d}\psi][\mathrm{d}\bar{\psi}] e^{-S_\psi} = \det(D)}$$

where the last equality is formal because the expression is divergent and needs regularizing. In fact we need two regulators:

- An infrared cutoff $\boxed{\mu}$ in order to have a discrete spectrum
- An ultraviolet cutoff $\boxed{\Lambda}$ in order to tame the short distance infinities

We will regularize the theory in the ultraviolet using a procedure introduced by A. Andrianov and Bonora (see also V. Andrianov, Novozhilov^2, Vassilevich)

The energy cutoff is enforced by considering only the first $\boxed{N}$ eigenvalues of $\boxed{D}$

Consider the projector $\boxed{P_N = \sum_{n=0}^{N} |\lambda_n\rangle \langle\lambda_n|}$ with $\boxed{\lambda_n}$ and $\boxed{|\lambda_n\rangle}$ the eigenvalues and eigenvectors of $\boxed{D}$

The integer $\boxed{N}$ is a function of the cutoff and is defined as

$$\boxed{N = \max n \text{ such that } \lambda_n \leq \Lambda}$$

We effectively use the N^{th} eigenvalue as cutoff

This is a purely algebraic cutoff does not refer to an underlying spacetime, it only refer to a Dirac operator, and could in principle be generalized to any wave function operator (KG, higher spin . . .)

The quantity $\boxed{N}$, and its behaviour as $\boxed{\Lambda}$ increase, carry information on the number of dimensions.

$$\boxed{N \sim \left(\frac{\Lambda}{\mu}\right)^d}$$

Although $\boxed{N}$ depends also on $\boxed{\mu}$ we will not discuss infrared issues in this talk.

The fermionic action is still invariant under the scale transformation

However the measure of the partition function is not invariant. This means that we have an anomaly.

The anomaly can however be cancelled adding another term to the action which will correct the measure. This term contains the bosonic degrees of freedom

Thus the cancellation of the anomaly forces the addition of the bosonic degrees of freedom to the fermionic one.

This new effective action is product of quantization, but in the end we will consider the full action to be the sum of two terms.

Define the created the regularization of the measure partition function

$$\boxed{Z_\Lambda(D) = \prod_{n=0}^{N} \lambda_n = \det\left(\mathbb{1} - P_N + P_N \frac{D}{\Lambda} P_N\right)}$$

$\boxed{Z_\Lambda}$ has a well defined meaning setting $\boxed{\psi = \sum a_n \left|\lambda_n\right\rangle \ \ , \bar{\psi} = \sum b_n \left|\lambda_n\right\rangle}$ with $\boxed{a_n, b_n}$ anticommuting (Grassman) quantities, we have

$$\boxed{Z_\Lambda(D) = \int \prod_{n=0}^{N} \mathrm{d}a_n \mathrm{d}b_n \mathrm{e}^{-\sum_{n=0}^{N} b_n \frac{\lambda_n}{\Lambda} a_n} = \det\left(D_N\right)}$$

where we defined $\boxed{D_N = 1 - P_N + P_N \frac{D}{\Lambda} P_N}$ which corresponds to set to $\boxed{1}$ all eigenvalues larger than $\boxed{1}$.

$\boxed{D_N}$ is dimensionless and depends on $\boxed{\Lambda}$ both explicitly and intrinsically via the dependence of $\boxed{N}$ and $\boxed{P_N}$

The compensating term, the effective action, is

$$\boxed{Z_{\mathrm{inv}\Lambda}(D) = Z_\Lambda(D) \int \mathrm{d}\phi\, \mathrm{e}^{-S_{\mathrm{anom}}}}$$

The calculation is standard and not difficult:

Define

$$Z^{-1}_{\text{inv}\Lambda}(D) = \int \mathrm{d}\phi Z_\Lambda^{-1}(\mathrm{e}^{-\frac{1}{2}\phi} D \mathrm{e}^{-\frac{1}{2}\phi})$$

therefore

$$S_{\text{anom}} = \log Z_\Lambda(D) Z^{-1}_{\text{inv}\,N}(D)$$

Let us indicate

$$Z_t = Z_\Lambda(\mathrm{e}^{-\frac{t}{2}\phi} D \mathrm{e}^{-\frac{t}{2}\phi})$$

therefore $Z_0 = Z_\Lambda(D)$ and

$$Z_\Lambda(D) Z^{-1}_{\text{inv}\,N}(D) = \int \mathrm{d}\phi \frac{Z_0}{Z_1}$$

and hence

$$S_{\text{eff}} = \int_0^1 \mathrm{d}t \partial_t \log Z_t = \int_0^1 \mathrm{d}t \frac{\partial_t Z_t}{Z_t}$$

We have the following relation that can easily proven:

$$D_N^{-1} = (1 - P_N + P_N D P_N)^{-1} = 1 - P_N + P_N D^{-1} P_N$$

and

$$\begin{aligned}
\partial_t Z_t &= \partial_t \det(\mathrm{e}^{-\frac{t}{2}\phi} D \mathrm{e}^{-\frac{t}{2}\phi})_N \\
&= \partial_t \mathrm{e}^{\mathrm{tr}\,\log(1-P_N+\mathrm{e}^{-\frac{t}{2}\phi} D_N \mathrm{e}^{-\frac{t}{2}\phi})} \\
&= \mathrm{Tr}\,(\partial_t \log(1 - P_N + \mathrm{e}^{-\frac{t}{2}} D_N \mathrm{e}^{-\frac{t}{2}\phi})) Z_t \\
&= \mathrm{Tr}\,((1 - P_N + \mathrm{e}^{-\frac{t}{2}\phi} D_N \mathrm{e}^{-\frac{t}{2}\phi})^{-1} \phi \mathrm{e}^{-\frac{t}{2}\phi} D_N \mathrm{e}^{-\frac{t}{2}\phi}) Z_t \\
&= \phi Z_t \,\mathrm{tr}\, P_N
\end{aligned}$$

In the end

$$\boxed{S_{\text{anom}} = \int_0^1 \mathrm{d}t \, \phi \, \mathrm{tr}\, P_N}$$

The quantity $\boxed{\mathrm{tr}\, P_N = N}$ is the number of eigenvalues of $\boxed{D}$ smaller than $\boxed{\Lambda}$. It depends on $\boxed{D_0}$ as well as its fluctuations in both the gravitational and gauge sectors, in fact we can express it as

$$\boxed{\mathrm{Tr}\, P_N = N = \mathrm{Tr}\, \chi\left(\frac{D_\phi^2}{\Lambda^2}\right)}$$

Where $\boxed{\chi}$ is the characteristic function of the interval $\boxed{[0,1]}$ and $\boxed{D_\phi = \mathrm{e}^{-\frac{1}{2}\phi} D \mathrm{e}^{-\frac{1}{2}\phi}}$

For $\boxed{\phi = 0}$ this is the Chamseddine-Connes **Spectral Action** introduced to describe the bosonic degree of freedom of the standard model coupled with gravity

We have obtained it as an additional term to the fermionic action upon quantizing it and demanding freedom from anomalies

The fact that the two terms of the spectral action must be on the same footing has been advocated already by Sitarz

To obtain the standard model take as algebra the product of the algebra of functions on spacetime times a finite dimensional matrix algebra

$$\boxed{\mathcal{A} = C(\mathbb{R}^4) \otimes \mathcal{A}_F}$$

Likewise the Hilbert space is the product of fermions times a finite dimensional space which contains all matter degrees of freedom, and also the Dirac operator contains a continuous part and a discrete one

$$\boxed{\mathcal{H} = \mathrm{Sp}(\mathbb{R}^4) \otimes \mathcal{H}_F}$$

$$\boxed{D_0 = \gamma^\mu \partial_\mu \otimes \mathbb{I} + \gamma \otimes D_F}$$

In its most recent form (Chamseddine-Connes-Marcolli) a crucial role is played by the mathematical requirements that the non-commutative algebra satisfies the requirements to be a manifold

Then the internal algebra, is almost uniquely derived to be

$$\boxed{\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})}$$

The bosonic spectral action can be evaluated using Vassilevich's manual on heath kernel techniques and the final result gives the action of the standard model coupled with gravity.

The fascinating aspect of this theory is that the Higgs appears naturally as the "vector" boson of the internal noncommutative degrees of freedom Connes, Lott, Dubois-Violette, Madore, Kerner . . . . In the process of writing the action all masses and coupling are used as inputs, but one saves one parameter.

The Higgs mass is predicted, in the present form of the model, to be $\boxed{\sim 170\text{GeV}}$. A value too small and experimentally disfavoured. Nevertheless I still find it fascinating that a theory without so little input finds a Higgs mass relatively close to the expected value

Technically the bosonic spectral action is a sum of residues and can be expanded in a power series in terms of Λ^{-1} as

$$S_B = \sum_n f_n \, a_n(D^2/\Lambda^2)$$

where the f_n are the momenta of χ

$$\begin{aligned} f_0 &= \int_0^\infty \mathrm{d}x \, x\chi(x) \\ f_2 &= \int_0^\infty \mathrm{d}x \, \chi(x) \\ f_{2n+4} &= (-1)^n \partial_x^n \chi(x)\Big|_{x=0} \quad n \geq 0 \end{aligned}$$

the a_n are the Seeley-de Witt coefficients which vanish for n odd. For D^2 of the form

$$D^2 = g^{\mu\nu}\partial_\mu\partial_\nu \mathbb{1} + \alpha^\mu \partial_\mu + \beta$$

defining

$$
\begin{aligned}
\omega_\mu &= \frac{1}{2} g_{\mu\nu} \left(\alpha^\nu + g^{\sigma\rho} \Gamma^\nu_{\sigma\rho} \mathbb{1} \right) \\
\Omega_{\mu\nu} &= \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu] \\
E &= \beta - g^{\mu\nu} \left(\partial_\mu \omega_\nu + \omega_\mu \omega_\nu - \Gamma^\rho_{\mu\nu} \omega_\rho \right)
\end{aligned}
$$

then

$$
\begin{aligned}
a_0 &= \frac{\Lambda^4}{16\pi^2} \int \mathrm{d}x^4 \sqrt{g}\, \mathrm{tr}\, \mathbb{1}_F \\
a_2 &= \frac{\Lambda^2}{16\pi^2} \int \mathrm{d}x^4 \sqrt{g}\, \mathrm{tr} \left(-\frac{R}{6} + E \right) \\
a_4 &= \frac{1}{16\pi^2} \frac{1}{360} \int \mathrm{d}x^4 \sqrt{g}\, \mathrm{tr} \, (-12 \nabla^\mu \nabla_\mu R + 5R^2 - 2 R_{\mu\nu} R^{\mu\nu} \\
&\quad +2 R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} - 60 RE + 180 E^2 + 60 \nabla^\mu \nabla_\mu E + 30 \Omega_{\mu\nu} \Omega^{\mu\nu})
\end{aligned}
$$

tr is the trace over the inner indices of the finite algebra $\mathcal{A}_F$ and in Ω and E are contained the gauge degrees of freedom including the gauge stress energy tensors and the Higgs, which is given by the inner fluctuations of D

The spectral action is growing to be "mature" for phenomenological predictions. Nevertheless we are "not yet there".

Apart from the Higgs prediction the cosmological scenario it gives, with the square of the curvature terms, may be of use in cosmology

There are other problems with this approach, it is Euclidean, fine tuning is needed, the coupling constants all meet in one point the quantization is done in the "commutative" way, which is somehow anticlimactic.

The fact that the model is "ad hoc" and in the end it writes a known action however is not a problem. The programme was to fit the standard model into a more general framework, not to derive it form an higher theory

For $\boxed{\phi}$ constant, after performing the integration we find

$$\boxed{S_{\text{anom}} = \int_0^{\phi} \mathrm{d}t' \sum_n \mathrm{e}^{(4-n)t'} a_n f_n = \frac{1}{8}(e^{4\phi} - 1)a_0 + \frac{1}{2}(e^{2\phi} - 1)a_2 + \phi a_4.}$$

There are just some numerical corrections to the first two Seeley-de Witt coefficients due to the integration in $\boxed{t\phi}$

I will not discuss in detail the case of $\boxed{\phi}$ nonconstant, in this case we have a dilaton in the theory which will couple with the other fields

The spectral action in this case involves

$$\boxed{D_\phi = \mathrm{e}^{-\phi}\gamma^\mu \left(\partial_\mu + A_\mu - \frac{1}{2}\partial_\mu \phi\right)}$$

where with A_μ I have generically indicated the gauge and spin connections. The calculation (still in progress).I can anticipate that it give a realistic effective potential which can be used to drive inflation.

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On the seventh he rested, while the universe was inflating