Noncommutative Geometry

Review of recent developments for non specialists

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In this talk I will present a review of recent developments in Noncommutative Geometry aimed at those who are not active in the field, apologies to the experts

I will of course use my personal preferences, prejudices, (lack of) expertise to choose which topics to threat in more or less detail

I will also concentrate on the physical applications of the theory, mostly leaving aside the rich mathematical developments.

I will not give the original references but will refer to some reviews

So, what were we talking about?

Unless you are lucky enough to be too young to having been around ten years ago, you will remember a famous article of Seiberg and Witten called String theory and noncommutative geometry

This article, which is Witten's second most cited article according to Spires, stimulated an enormous amount of interest in Noncommutative Geometry which became a hot topic for a couple of years. To the extent that there are still people considering it a part of string theory.

Then the interest in the string community diminished as other interesting stringy topics became hot, died, other topics came and went

There were people doing noncommutative geometry before, and there is still a sizeable community still doing it, although the link with string people has been lost (fortunate? a pity?)

Undeniably however the connection with string theory gave a strong impulse to the study of noncommutative geometry and to field theories on noncommutative spaces in particular

So, what is Noncommutative Geometry?

The answer: "it is the theory for which $\boxed{[x,y]=i\theta}$" is extremely reductive. Like saying that quantum mechanics is the theory for which the exchanges of energy are discrete.

I prefer to say see noncommutative geometry as a bag of tools which describes geometry not as a set of points, lines, vectors etc. but rather using the "functions" defined on it. And the tools are sufficiently flexible to be applied also in cases for which it does not make sense to talk of points of the space. In this case the functions become objects multiplied in a noncommutative guise.

In this respect the quantum phase space is the master example. Due to the uncertainty principle we cannot speak of points on phase space, and we are led to the description of observables as noncommuting operators

In mathematics there are several interesting examples of noncommutative spaces which have nothing to do with noncommuting coordinates

Several of the recent applications of noncommutative geometry concern the possibility that spacetime at the Planck length is described by a different form of geometry

There are several reasons, going back to Bronstein in the 30's to argue that in a theory of quantum gravity it would be impossible to measure the position of points with Planck length accuracy, because a black hole would otherwise be formed.

This is quite natural in a theory like string theory in which space-time is a derived concept (a field), and in fact in the vertex operators of an strings reproduce (in the presence of a background field and in a particular limit) a noncommutative geometry of the kind $\boxed{[x^i, x^j] = i\theta^{ij}}$ which in turn is described, at the level of fields and function, by the Grönewold-Moyal product

$$\boxed{(f \star g)(x) = f e^{\frac{i\theta^{ij}}{2}\overleftarrow{\partial_i}\overrightarrow{\partial_j} - \overleftarrow{\partial_j}\overrightarrow{\partial_i}} g = fg + \frac{i\theta^{ij}}{2}(\partial_i f \partial_j g - \partial_j f \partial_i g) + O(\theta^2)}$$

This has led to the study of field and gauge theories on non-commutative spaces in which the ordinary product among fields is substituted by this $\boxed{\star}$-product

So, what else Noncommutative Geometry?

There have been historically some aspects of noncommutative geometry notable for their physical applications

- Noncommutative field theory
- Connes' approach to the standard model
- Fuzzy spaces

And then there has been the study of quantum groups

I will mainly concentrate on the field theory (with a mention of quantum groups) but first let me say a few words on fuzzy spaces and standard model.

In Connes' approach the standard model is aimed at understanding the "geometry" of it

The idea is again that all information about a physical system is contained in the algebra of functions (spacetime) represented as operators on a Hilbert space (states), with the action and metric properties encoded in a generalized Dirac operator

The game is then just to see which sets of data (an algebra, a Hilbert space, a Dirac operator) reproduce the standard model

For example if the spacetime is two copies of a manifold the generalization of the electrodynamics action gives a $\boxed{U(1) \times U(1) \to U(1)}$ Higgs mechanism

Instead for an algebra given by functions on spacetime with values in $\boxed{\mathbb{C} \otimes \mathbb{H} \otimes M_3}$, (complex numbers, quaternions, three by three matrices) we obtain the standard model

The action is purely based on spectral properties of a covariant Dirac operator

The fascinating aspect of this theory is that the Higgs appears naturally as the "vector" boson of the internal noncommutative degrees of freedom. In the process of writing the action all masses and coupling are used as inputs, but one saves one parameter, which can be the Higgs mass

There are problems with this approach as well, mainly the fact that it is naturally classic, and that the quantization is done in the "commutative" way, which is somehow anticlimactic.

Recently work by Chamseddine, Connes and Marcolli several of the earliest problems (neutrino masses, fermion doubling) have been solved, but the model is growing in complexity, and is not particularly easy to manage. Still I feel the original idea remains an excellent one

Review: Chamseddine, Connes arXiv:0812.0165

Fuzzy spaces are finite (matrix) approximations to spaces which however, unlike ordinary lattices, maintain the symmetries of the original spaces, at the price of having a noncommutative product among functions

The prototype is the two-sphere described by noncommuting coordinates $\boxed{[x_i, x_j] = \kappa \varepsilon_{ijk} x_k}$, but there are fuzzy approximations to other spheres, projective spaces and the disc, and supersymmetric versions

Recently some groups have started to do actual calculation of field theory on these fuzzy spaces, finding the phase transitions. While there has been no phenomenological application yet, the tool is being developed for actual calculations and we are not too far from it

Review: Balachandran Kurkcuoglu Vaidya, Lectures on fuzzy and fuzzy SUSY physics. hep-th/0511114

Field and gauge theories with the Moyal product are by far the most studied aspects of noncommutative geometry

See Szabo Quantum field theory on noncommutative spaces Physics Report hep-th/0109162

The prototype of this is a $\boxed{\phi^4}$ theory described by the action

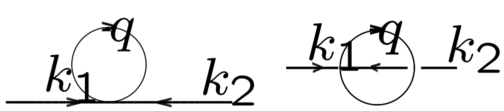
$$\boxed{S_\star = \int d^dx \partial_\mu\varphi \star \partial^\mu\varphi + m^2 \varphi \star \varphi + \frac{g^2}{4!}\varphi \star \varphi \star \varphi \star \varphi}$$

Apart from the mentioned string theory, one of the hopes for the study of noncommutative field theory was that the presence of the cutoff $\boxed{\theta}$ would regularize the theory (in analogy withe $\boxed{\hbar}$ and the ultraviolet catastrophe of the black body)

The free theory is unchanged because of the integral property $\boxed{\int d^dx f \star g = \int d^dx f g}$. But the vertex gets a phase.

$$\boxed{V = (2\pi)^4 g \delta^4 \left(\sum_{a=1}^{4} k_a \right) \prod_{a<b} \mathrm{e}^{-\frac{i}{2}\theta^{\mu\nu} k_{a\mu} k_{b\nu}}}$$

The vertex is not anymore invariant for exchange of the momenta (only for cyclic permutations), and causes a difference between planar and nonplanar diagrams



This is the Ultraviolet/Infrared Mixing. The phenomenon for which some ultraviolet divergences disappear, just to reappear as infrared divergences.

This can heuristically seen as a consequence of the generalized uncertainty principle. the short distance behaviour in coordinate $\boxed{x^i}$ is linked to the long distance behaviour in $\boxed{\theta_{ij}x^j}$

At any rate not all divergences are suppressed (planar diagrams are unchanged)

The strategy can now be to either change the action or change the product

There are two known actions which are renormalizable at all orders

Review Rivasseau Non-commutative Renormalization arXiv:0705.0705

There are two actions which can be shown to be renormalizable to all orders. They are corrections to the earlier $\boxed{S_\star}$

$$\boxed{S_{GW} = S_\star + \int dx \frac{\Omega^2}{2}(\theta^{-1}_{ij}x^j\varphi) \star (\theta^{-1\,ik}x_k\varphi)}$$

$$\boxed{S_{Paris} = S_\star + \int dx \ \alpha\varphi \star \frac{1}{\theta^2\Box}\varphi}$$

Although the two models above do not look particularly "natural", they provide the first examples of a nonperturbatively renormalizable interacting field theory without Landau ghost. The dream of constructive field theory.

Another possibility is to change the $\boxed{\star}$ product and keep the same action

It is possible to prove that any associative translationally invariant product gives the same structure of Ultraviolet/Infrared mixing as the one of the Moyal product

No matter how complicated is the product, if it is translationally invariant $\boxed{[x^i, x^j] = \theta^{ij}}$ is constant

Then, although the Green's functions and also the propagators may be different, the one loop contribution to the propagator will always be of the same type as in Moyal

Ultraviolet/Infrared mixing seems to be intimately connected with the structure of spacetime, and although it may be corrected by extra terms in the action, it carries information about the pointless nature of noncommutativity

Even if translationally invariant a field theory based on the Moyal product is not Lorentz invariant

In four dimension $\boxed{\theta^{ij}}$ is not a scalar and naively it should transform under rotations and boost, and picks up two preferred directions in space

It turns out that a field theory with the Moyal product is has a quantum symmetry of a particular kind, obtained by a Drinfeld twist

Review: Aschieri, Dimitrijevic, Kulish, Lizzi, Wess
"Noncommutative Spacetimes: Symmetries in Noncommutative Geometry and Field Theory",
Springer Lecture Notes **ON SALE APRIL 23!!!**

Consider the usual action of the Lie algebra $\boxed{L}$ of differential operators on the algebra $\boxed{\mathcal{A}}$ of functions with the usual commutative product

The usual product can be seen as a map from $\boxed{\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}}$

$\boxed{m_0(f \otimes g) = fg}$ pointwise multiplication

The Leibnitz rule imposes a coalgebra structure of the Lie algebra:

$$\boxed{M(fg) = M(f)g + fM(g) = m_0(\Delta(M)(f \otimes g))}$$

where $\boxed{M}$ is a generic element of the Lie algebra, and

$$\boxed{\Delta : L \to L \otimes L}$$

$$\boxed{\Delta(M) = M \otimes 1 + 1 \otimes M}$$

Consider the Moyal product comes can be seen as a Twisted product

$$\boxed{(f \star g)(x) = m_0[\mathcal{F}^{-1} f \otimes g] \equiv m_\theta[f \otimes g]}$$

where

$$\boxed{m_0(f \otimes g) = fg}$$

is the ordinary product and

$$\boxed{\mathcal{F} = e^{\frac{i}{2}\theta^{ij}\partial_{x_i}\otimes\partial_{y_j}}}$$

is called the twist.

The Noncommutative space is obtained twisting the tensor product, and using the ordinary product.

We have to then revise the Leibnitz rule:

$$\boxed{M(f \star g) = m_\theta \Delta_\theta (f \otimes g) = m_0 \Delta M(\mathcal{F}^{-1}(f \otimes g))}$$

where

$$\boxed{\Delta_\theta = \mathcal{F} \Delta \mathcal{F}^{-1}}$$

The algebra structure remains unchanged, what changes is the coalgebra structure, that is the way to "put together representations".

counit and antipode remain unchanged.

We still have a symmetry, but it is a twisted and we have to understand the meaning

Since the Lie algebra structure remains the same we can still talk of the usual particles (Wigner representation)

Take the point of view to twist with the above operator, in its proper representation, all products encountered

Given a generic product from the product of two spaces into a third

$$\boxed{\mu : \ X \times Y \longrightarrow Z}$$

we associate a deformed product

$$\boxed{\mu_\star : \ \mu \circ \mathcal{F}^{-1}(X \otimes Y) \longrightarrow Z}$$

In particular when $\boxed{X = Y = Z = C(M)}$, the algebra of functions on a manifold, the usual pointwise product is deformed in the appropriate $\boxed{\star}$ product

The presence of a symmetry, even if a quantum one, is always a good guidance, and it has been used for S-matrix, gravity, spin statistics, phenomenology, and for the construction and application to other noncommutative geometries

As an example I will describe very briefly how a coherent twisting procedure makes the S-matrix of Moyal and the Wick-Voros products equivalent

Introduce the Wick-Voros:

$$\boxed{z_\pm = \frac{x^1 \pm i x^2}{\sqrt{2}}}$$

$$\boxed{\partial_\pm = \frac{\partial}{\partial z_\pm} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^1} \mp i \frac{\partial}{\partial x^2}\right)}$$

$$\boxed{f \star_V g = \sum_n \left(\frac{\theta^n}{n!}\right) \partial_+^n f \partial_-^n g = f \mathrm{e}^{\theta \overleftarrow{\partial_+} \overrightarrow{\partial_-}} g}$$

The two products are equivalent in the sense that there is an invertible map $\boxed{T = \mathrm{e}^{\frac{\theta}{4}\nabla^2}}$ with the property

$$\boxed{T(f \star_{\mathsf{M}} g) = T(f) \star_{\mathsf{V}} T(g)}$$

Nevertheless the four points greens functions are different

$$\boxed{G_0^{(4)} = \frac{\mathrm{e}^{\sum_{a\le b} k_a \bullet k_b}}{\prod_{a=1}^{4}\left(k_a^2 - m^2\right)}\delta\left(\sum_{a=1}^{4} k_a\right)}$$

where

$$\boxed{k_a \bullet k_b = \begin{cases} -\frac{i}{2}\theta^{ij}k_{ai}k_{bj} & \text{Moyal} \\ \\ -\theta k_{a-}k_{b+} = -\frac{1}{2}\left(\theta k_{ai}{k_b}^i + i\theta^{ij}k_{ai}k_{bj}\right) & \text{Wick-Voros} \end{cases}}$$

The Green's functions are therefore different

The S-matrices in the two cases are however the same, if one twists *all* the products involved (and proper recognition of the asymptotic states):

1. The tensor product in the two particle state

$$\boxed{\tilde{\mathcal{F}}_\star^{-1} |k_a\rangle \otimes |k_b\rangle = |k_a, k_b\rangle_\star = \mathrm{e}^{-\frac{i}{2}\theta^{ij} k_{a_i} \otimes k_{b_j}} |k_a\rangle \otimes |k_b\rangle = a^\dagger_{k_a} \star a^\dagger_{k_b} |0\rangle}$$

2. The inner product

$$\boxed{\left\langle \cdot \overset{\star}{\Big|} \cdot \right\rangle : |k\rangle \otimes |k'\rangle \longrightarrow \langle \cdot | \cdot \rangle \circ \mathcal{F}^{-1} : |k\rangle \otimes |k'\rangle = \tilde{\mathcal{F}}^{-1}(k, k') \langle k | k' \rangle}$$

$$\boxed{= \langle 0 | a_k \star a^\dagger_{k'} |0\rangle}$$

3. Inner product among two-particle states

$$\boxed{\left\langle k_1 k_2 \overset{\star}{\Big|} k_3 k_4 \right\rangle = \langle \cdot | \cdot \rangle \circ \Delta_\star(\mathcal{F}^{-1})(\,|k_1 k_2\rangle \otimes |k_3 k_4\rangle)}$$

where by $\boxed{\Delta_\star(\mathcal{F}^{-1})}$ is defined by $\boxed{\Delta(\partial \otimes \partial) = \Delta(\partial) \otimes \Delta(\partial)}$

$$\boxed{\left\langle k_1, k_2 \overset{\star\mathsf{M}}{\Big|} k_3, k_4 \right\rangle = \mathrm{e}^{\frac{i}{2}\theta^{ij}(k_{1_i}+k_{2_i})(k_{3_j}+k_{4_j})} \langle k_1, k_2 | \, k_3, k_4 \rangle}$$

$$\boxed{\left\langle k_1, k_2 \overset{\star\mathsf{V}}{\Big|} k_3, k_4 \right\rangle = \mathrm{e}^{\theta(k_{1-}+k_{2-})(k_{3+}+k_{4-})} \langle k_1, k_2 | \, k_3, k_4 \rangle}$$

Putting everything together

$$\boxed{{}_\star\left\langle k_1, k_2 \overset{\star}{\Big|} k_3, k_4 \right\rangle_\star = \langle 0|\, a_{k_1} \star a_{k_2} \star a^\dagger_{k_3} \star a^\dagger_{k_4} \,|0\rangle}$$

$$\boxed{= \mathrm{e}^{-\sum_{a<b} k_a \bullet k_b} \langle k_1, k_2 |\, k_3, k_4 \rangle}$$

Only if all these twistings are made we find that the two S-matrices are the same as expected

I think however that we still have to learn completely how to "use" the quantum symmetry for physical applications.

The main challenge being the multiparticle states, since the "putting together" of representations is governed by the coproduct.

All these aspects of noncommutative field theories are now getting to be mature for phenomenological applications, and actual physical prediction. The most promising of which are for cosmology

Sorry no review yet, but look for example at Karwan 0903.2906 for cosmology and the forthcoming lectures of Gracia-Bondìa on the proceedings of the Holbaeck conference for the gravity part

But, are we using the right kind of noncommutativity?

There is another kind on noncommutative space, connected to a quantum symmetry which has always attracted attention: $\boxed{\kappa}$-Minkowski

$$\boxed{[x_0, x_i] = \frac{i}{\kappa} x_i \quad , \quad [x_i, x_j] = 0}$$

This is the homogenous space of a deformed symmetry (called $\boxed{\kappa}$-Poincaré for which not only the coproduct, but also the commutator is deformed

The presence of deformed commutation relations brings different Casimir functions, deformations of $\boxed{m^2 = e^2 - P^2}$, and a deformation of the dispersion relations

Unfortunately the new commutation relations are non linear, and therefore one must be careful on which basis to use

Field and gauge theories based on $\boxed{\kappa}$-Minkowski are being built. Here the problem is that we have a quantum symmetry that deforms also translations. Nevertheless there is activity in this area

Of extreme interest is the possibility to have gravity as an emergent phenomenon from a matrix model obtained from noncommutative geometry. In a nutshell this is the idea

Start from a matrix model in $\boxed{d}$ dimensions with a flat metric $\boxed{g_{ab}}$

$$\boxed{S = -\,\mathrm{Tr}\,[X^a, X^b][X^c, X^d] g_{ac} g_{bd}}$$

with equations of the motion

$$\boxed{[X^a, [X^b, X^c]] g_{ab} = 0}$$

with a solution (not the most general) which is basically the one of the Moyal product ($\boxed{\theta}$ a number, not a matrix in the internal indices)

$$\boxed{[Y_0^a, Y_0^b] = \theta_0^{ab}}$$

A more interesting solution is

$$\boxed{X_0^a = Y_0^a \otimes \mathbb{I}_n}$$

consider now small fluctuations

$$\boxed{X^a = X_0^a + \mathcal{A}^a(Y)}$$

where $\boxed{\mathcal{A}^a(Y) = A_0^a + A_\alpha \lambda^\alpha}$ is a matrix valued function which split into a trace part and a traceless part, where $\boxed{\lambda^\alpha}$ are the generators of $\boxed{SU(n)}$. Make now a different splitting, in which you separate the traceless part

$$\boxed{X^a = Y^a + A_\alpha \lambda^\alpha}$$

but now we have nonconstant commutativity for the "background"

$$\boxed{[Y^a, Y^b] = \theta^{ab}(Y)}$$

We now couple a scalar field $\boxed{\Phi}$ to the theory coupled with the action

$$\boxed{S_\phi = -\,\mathsf{Tr}\,[X^a,\Phi][X^b,\Phi]g_{ab}}$$

we have that since we are interpreting the $\boxed{X}$'s as generalized noncommutative coordinates, which in a "commutative limit" gives

$$\boxed{[X^a,\Phi] \sim \theta^{ab}\partial_a\Phi + [A^a,\Phi]}$$

so that the action becomes

$$\boxed{S_\phi = -\,\mathsf{Tr}\,\theta^{ab}\theta^{cd}g_{ac}(\partial_b\Phi + [A_c,\Phi])(\partial_d\Phi + [A_d,\Phi]) = -\,\mathsf{Tr}\,G^{ab}(Y)D_a\Phi D_b\Phi}$$

$$\boxed{G^{ab} = \theta^{ac}\theta^{bd}g_{cd}}$$

$$\boxed{D_a\Phi = \partial_a\Phi + [A_a,\Phi]}$$

In this way we have that gravity emerges in a low energy (commutative) approximation of a matrix theory. This is turn would be an approximation of a more general noncommutative space described by a noncommutative algebra with a representation as operator (infinite matrices) on some Hilbert space

It is possible to couple fermions to the theory, consider gauge theory and connect with the matrix models coming from branes

Review: Steinacker arXiv:0712.3194

So, should we keep on doing Noncommutative geometry?

I do not know about you, but I think that the structure of spacetime at the Planck scale is one of the fundamental problems in today's physics.

Studying noncommutative geometry we are learning a lot, and while it may be likely that we have not yet hit on the right structure, I believe that the correct theory will need to use the tools and the idea of noncommutative geometry

There has been already a connection between strings and noncommutative geometry, and lately also connections with loop quantum gravity are emerging

At the same time we are finally getting, certainly through astroparticle and cosmology, and possibly LHC, inputs of higher energy physics. The totally novel "Planckish" may not be so far away, and it may be described by a new spacetime. Which is worth investigating