Deformation Quantization, Monopoles and Quaternions

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In this talk I wish to build a deformed $\boxed{\star}$ algebra with the Poisson bracket of a particle in the presence of a magnetic monopole

The magnetic monopole is one of the most studied particle in physics, pity it does not exist!

At least no one has seen one and the usual wisdom is that this kind of purely electromagnetic monopole is not likely to be there. Other kind (non abelian) monopoles may play a role in cosmology, Nevertheless magnetic monopoles are certainly a source of interest, and fun in mathematics and physics

We found (following Emch and Jadczyk) that quaternions give useful and amusing structure with which say yet more things on this chapter of mathematical physics

The great interest of the magnetic monopole is that it is archetypical of many interesting mathematical structures, and the quantization of a particle in the presence of monopole is a most interesting one

It is impossible to define globally a potential (a connection) as the corresponding fiber bundle is not trivial, and therefore one has to define local charts

The alternative is to introduce the monopole via a noncanonical Poisson Structure

I will describe the interaction of a particle in a monopole field with the introduction of the following Poisson Structure

$$\boxed{\{x^i, x^j\} = 0}$$

$$\boxed{\{x^i, p_j\} = \delta^i_j}$$

$$\boxed{\{p_i, p_j\} = \frac{1}{2}\varepsilon_{ijk}\frac{x^k}{|x|^3}}$$

Where the monopole is in the origin and I have normalized its charge to 1

This gives a symplectic structure

$$\boxed{dx^i dp_i + \frac{1}{2}\varepsilon_{ijk}\frac{x^i}{|x|}dx^i dx^j}$$

$$\boxed{\omega(\eta,\xi,\eta',\xi') = \eta\cdot\xi' - \eta'\cdot\xi + \frac{1}{2}\varepsilon^{ijk}\frac{x^i}{|x|}\eta^j {\eta'}^k}$$

with $\boxed{(\xi\wedge\xi')_i = \varepsilon_{ijk}\xi_j\xi_k}$

We therefore need $\boxed{\star}$ product on $\boxed{\mathbb{R}^3\times(\mathbb{R}^3\backslash\{0\})}$ which reproduces this Poisson structure (up to an $\boxed{\mathrm{i}}$) as the commutators for the coordinates.

In the canonical case, i.e. had the monopole not being there, I would built the product via Weyl quantization. Let me show this step by step

First I consider the Fourier transform of the functions

$$\boxed{\tilde{f}(\eta,\xi) = \frac{1}{\sqrt{2\pi}^3}\int \mathrm{d}x\mathrm{d}p\,\mathrm{e}^{-\mathrm{i}(\eta\cdot p+\xi\cdot x)} f(x,p)}$$

Then I perform some sort of inverse Fourier transform but instead of $\boxed{\mathrm{e}^{-\mathrm{i}(\eta\cdot p+\xi\cdot x)}}$ I insert an operator $\boxed{\widehat{T}(\eta,\xi)}$. In this way to a function $\boxed{f}$ I associate an operator $\boxed{\widehat{\mathcal{W}}(f)}$

$$\boxed{\widehat{f} = \widehat{\mathcal{W}}(f) = \frac{1}{\sqrt{2\pi}^3}\int \mathrm{d}\eta\mathrm{d}\xi\,\widehat{T}(\eta,\xi)\tilde{f}(\eta,\xi)}$$

Represented on the Hilbert space of square integrable functions of the $\boxed{x}$'s in terms of the usual $\boxed{\hat{X}^i\psi(x) = x^i\psi(x)}$ and $\boxed{\hat{X}_i\psi(x) = -\mathrm{i}\partial_i\psi(x)}$ we have

$$\boxed{T(\eta,\xi) = \mathrm{e}^{\mathrm{i}(\eta\cdot\hat{P}+\xi\cdot\hat{X})}}$$

The $\boxed{T}$'s form what is called a Weyl system

$$\boxed{T(\eta,\xi)T(\eta',\xi') = T(\eta+\eta',\xi+\xi')\mathrm{e}^{\frac{\mathrm{i}}{2}(\eta\cdot\xi'-\eta'\cdot\xi)}}$$

where in the exponential we can recognize the canonical symplectic structure

We identify the configuration space $\boxed{\mathbb{R}^3}$ as a Lagrangian subspace of $\boxed{\mathbb{R}^6}$. Considering square integrable functions on this lagrangian submanifold we can identify

$$\boxed{T(\eta, 0)\psi(x) = \psi(x + \eta)}$$

$$\boxed{T(0, \xi)\psi(x) = \mathrm{e}^{\mathrm{i}\xi \cdot x}\psi(x)}$$

As is known one can realize $\boxed{\mathbb{R}^6}$ as the cotangent space of $\boxed{\mathbb{R}^3}$ and the symplectic form is the differential of the Lioville 1-form over the Lagrangian submanifold

The Weyl map has an inverse (Wigner map) $\boxed{\mathcal{W}^{-1}}$ and I can express them in terms of a quantizer $\boxed{\Omega}$ and a dequantizer $\boxed{\Gamma}$:

$$\boxed{\hat{f} = \int \mathrm{d}x\mathrm{d}p\hat{\Omega}(x,p)f(x,p)}$$

$$\boxed{f(x,p) = \mathrm{Tr}\,\Gamma(x,p)\hat{f}}$$

For the canonical case $\boxed{\hat{\Omega} = \hat{\Gamma}}$

$$\boxed{\hat{\Omega}(x,p) = \frac{1}{\sqrt{2\pi}^3}\int \mathrm{d}\eta\mathrm{d}\xi\hat{T}(\eta,\xi)\mathrm{e}^{\mathrm{i}(\eta\cdot p+\xi\cdot x)}}$$

Then the $\boxed{\star}$ is defined as

$$\boxed{(f \star g)(x,p) = \mathcal{W}^{-1}\left(\widehat{\mathcal{W}}(f)\widehat{\mathcal{W}}(g)\right)}$$

$$\boxed{x^i \star p_j - p_j \star x^i = \mathrm{i}\delta^i_j}$$

And we have $\boxed{\star}$ quantized the canonical commutation relations

All this for the position dependent symplectic structure induced by the monopole does not work

We would like to build a quantization map, a quantizer, a Weyl system and a $\boxed{\star}$ product which reproduces the quantization of a particle in the presence of the monopole and its symplectic structure

We will do this with the help of quaternions and operators on a quaternionic Hilbert space (module)

We will rely heavily on the work of Emch and Jadczyk (see `quant-ph`/9803002) who introduced the monopoles in the quaternionic quantization

They were interested in quaternionic quantum mechanics, while we use quaternions solely as a mean to obtain a $\boxed{\star}$ product from complex valued functions to complex valued functions

Notations: we will indicate all quaternionic quantities in $\mathsf{sanserif}$. The three quaternionic complex quantities are called $\boxed{\mathsf{e}_i}$ and the identity is $\boxed{\mathsf{e}_0}$. A generic quaternion is $\boxed{\mathsf{q} = q^\mu \mathsf{e}_\mu}$. and we can define a trace $\boxed{\mathsf{tr}}$

The following hold:

$$\boxed{\mathsf{e}_i^\dagger = -\mathsf{e}_i \quad , \quad \mathsf{e}_0^\dagger = \mathsf{e}_0}$$

$$\boxed{\mathsf{e}_i \mathsf{e}_j = -\delta_{ij} \mathsf{e}_0 + \varepsilon_{ijk} e_k}$$

$$\boxed{\mathsf{tr}\,\mathsf{e}_0 = 2 \quad , \mathsf{tr}\,\mathsf{e}_i = 0}$$

The quaternionic Hilbert space is composed by functions on $\boxed{\mathbb{R}^3}$ with value in the quaternions and with the quaternionic valued inner product

$$\boxed{\langle\psi|\,\phi\rangle = \int \mathrm{d}x\,\psi^\dagger\phi}$$

Quaternionic scalar act on the right, operators act on the left. The position operator is as usual

$$\boxed{\widehat{\mathsf{x}}\psi(x) = x\mathsf{e}_0\psi(x)}$$

A generic quaternion such that $\boxed{\mathsf{q}^2 = -1}$ is called a imaginary unit

The three $\boxed{\mathsf{e}_i}$ are imaginary units, and they in some sense select the three orthogonal axis

Let me introduce a spherically symmetric quaternionic unit

$$\boxed{\mathsf{j}(x) = \frac{\mathsf{e} \cdot x}{|x|}}$$

and the corresponding quaternionic linear operator

$$\boxed{\hat{\mathsf{J}}\psi(x) = \mathsf{j}(x)\psi(x)}$$

With this quaternionic unit we can map a complex valued function to a quaternion valued function

$$\boxed{f(x,p) = f_r(x,p) + \mathsf{i} f_i(x,p) \rightarrow \mathsf{f}(x,p) = f_r(x,p)\mathsf{e}_0 + \mathsf{j}(x) f_i(x,p)}$$

with $\boxed{f_{r,i} \in \mathbb{R}}$ and the inverse map

$$\boxed{\mathsf{f}(x,p) \rightarrow \mathsf{tr}\,\mathsf{f}(x,p) - \mathsf{i}\,\mathsf{tr}\,\mathsf{j}(x)\mathsf{f}(x,p)}$$

This is an isomorphism between the algebra of complex valued functions and a commutative subalgebra of the quaternion valued functions

$\boxed{\hat{J}}$ commutes with $\boxed{\hat{x}}$ but does not commute with $\boxed{\partial_i}$. In fact we have to define a good which operator which would generate translation. Define the covariant derivative

$$\boxed{\hat{\nabla}_i = \partial_i + \frac{1}{2}\frac{\varepsilon_{ijk}\hat{x}^j e_k}{\|x\|^2}}$$

The covariant derivative commutes with $\boxed{\hat{J}}$ and it has the correct commutation with the position operator

$$\boxed{[\hat{\nabla}_i, \hat{x}^j] = e_0 \delta_i^j}$$

The geometric origin of this covariant derivative is the lift od the usual derivative to a $\boxed{\mathsf{SU}(2)}$ bundle.

Considering the representation of quaternions in terms of $\boxed{\mathsf{SU}(2)}$ Pauli matrices

$$\boxed{\mathsf{e}_i = -\mathsf{i}\sigma_i}$$

The quaternionic valued functions are sections of this bundle and we have the associated vector bundle with structure group $\boxed{\mathsf{SU}(2)}$.

Recall the usual Hopf fibration $\boxed{\pi : \mathrm{SU}(2) \to \mathbb{S}^2}$, then for $\boxed{s \in \mathrm{SU}(2)}$ and $\boxed{x \in \mathbb{R}^3 - \{0\}}$ we have the projection

$$\boxed{\pi(s) = s^{-1}\sigma_3 s = x^i \sigma_i}$$

and

$$\boxed{\sigma_i \mathrm{d}x^i = [s^{-1}\mathrm{d}s, x^i\sigma_i]}$$

The connection corresponding to the monopole gauge potential is

$$\boxed{A = \frac{1}{2}\frac{\varepsilon_{ijk}\mathrm{e}_i x^i \mathrm{d}x^k}{\|x\|^2} = \frac{1}{2}\frac{[\mathrm{e}\cdot x, \mathrm{e}\cdot \mathrm{d}x]}{\|x\|^2}}$$

The lift of the translations vector field $\boxed{u^i\partial_i}$ is

$$\boxed{\nabla_u = \mathrm{e}_0 u^i \partial_i + \frac{1}{2}\frac{[\mathrm{e}\cdot x, \mathrm{e}\cdot \mathrm{d}x]}{\|x\|^2}}$$

Define the momentum operator

$$\boxed{\hat{\mathsf{p}}_i = \hat{\mathsf{J}}\hat{\nabla}_i}$$

and the Hamiltonian $\boxed{\hat{\mathsf{H}} = \frac{1}{2m}\hat{\mathsf{p}}^2}$, then the operator $\boxed{\mathsf{e}^{\hat{\mathsf{J}}\hat{\mathsf{H}}t}}$ given the correct Heisenberg equations of the motion for a particle in the field of monopole:

$$\boxed{\partial_t^2 \hat{\mathsf{x}}^i = \frac{1}{2m}\varepsilon_{ijk}\left(\partial_t \hat{\mathsf{x}}^j \hat{\mathsf{B}}^k - \hat{\mathsf{B}}^x \partial_t \hat{\mathsf{x}}^j\right)}$$

with $\boxed{\hat{\mathsf{B}}}$ the operator corresponding to the magnetic field of monopole

$$\boxed{(\widehat{\mathsf{B}}^i\psi)(x) = \frac{1}{2}\frac{x^i}{\|x\|^3}\psi(x)}$$

This momentum operator has the correct commutation relations with the position operator:

$$\boxed{[\widehat{\mathsf{x}}_i, \widehat{\mathsf{p}}^j] = \widehat{\mathsf{J}}\delta_i^j}$$

$$\boxed{[\widehat{\mathsf{p}}_i, \widehat{\mathsf{p}}_j] = \widehat{\mathsf{J}}\frac{1}{2}\varepsilon_{ijk}\frac{\widehat{\mathsf{x}}^k}{|\widehat{\mathsf{x}}|^3}}$$

Since the $\boxed{\widehat{\mathsf{p}}}$ do not commute they cannot generate translations. In fact they do generate translations up to a phase

$$\boxed{\mathsf{e}^{\mathsf{j}\eta\cdot\widehat{\mathsf{p}}}\psi(x) = \mathsf{w}(\eta; x-\eta)\psi(x-\eta)}$$

where

$$\boxed{\mathsf{w}(\eta; x) = \frac{1}{\sqrt{2}}\left(\sqrt{1+\frac{\|x\|^2+\eta\cdot x}{\|x\|\,\|x+\eta\|}}+\mathsf{j}(x\wedge\eta)\sqrt{1-\frac{\|x\|^2+\eta\cdot x}{\|x\|\,\|x+\eta\|}}\right)}$$

$$\boxed{= \cos\left(\frac{\theta}{2}\right)+\mathsf{j}(x\wedge\eta)\sin\left(\frac{\theta}{2}\right)}$$

$\boxed{\theta}$ the angle between $\boxed{x}$ and $\boxed{\eta}$

There are some useful identities for $\boxed{\mathsf{w}}$

$$\boxed{\mathsf{w}(0;x) = 0}$$

$$\boxed{\mathsf{w}(\eta;x)\mathsf{w}^\dagger(\eta;x) = \mathsf{e}_0}$$

$$\boxed{\mathsf{w}(\eta;x-\eta) = \mathsf{w}^\dagger(-\eta;x)}$$

$$\boxed{\mathsf{w}(t\eta;x+s\eta)\mathsf{w}(s\eta;x) = \mathsf{w}((s+t)\eta;x)}$$

$$\boxed{\forall x,\eta \in \mathbb{R}^3,\ \ s,t \in \mathbb{R}}$$

In these identities is hidden the Dirac quantization. If the coefficient of the magnetic field is not an integer $\boxed{\mathsf{w}}$ ceases to be unitary

These generalized translations generate a projective representation of the group of translations

$$\boxed{\mathrm{e}^{\widehat{\mathrm{J}}\eta\cdot\widehat{\mathrm{p}}}\mathrm{e}^{\widehat{\mathrm{J}}\eta'\cdot\widehat{\mathrm{p}}} = \mathrm{e}^{\widehat{\mathrm{J}}(\eta+\eta')\cdot\widehat{\mathrm{p}}}\widehat{\mathrm{M}}(\eta,\eta')}$$

with

$$\boxed{\left(\widehat{\mathrm{M}}(\eta,\eta',x)\psi\right)(x) = \mathrm{w}^{\dagger}(\eta+\eta';x)\mathrm{w}(\eta;x+\eta')\mathrm{w}(\eta';x)\psi(x)}$$

Also $\boxed{\widehat{\mathrm{M}}}$ is unitary

This prepares us for the introduction of a rather natural operator

$$\boxed{\widehat{\mathsf{T}}(\eta,\xi) = \mathsf{e}^{\widehat{\mathsf{J}}(\eta\cdot\widehat{\mathsf{p}}+\xi\cdot\widehat{\mathsf{x}})}}$$

This operator closes a generalized monopole-Weyl system

$$\boxed{\widehat{\mathsf{T}}(\eta,\xi)\widehat{\mathsf{T}}(\eta',\xi') = \widehat{\mathsf{T}}(\eta+\eta',\xi+\xi')\widehat{\mathsf{M}}(\eta,\eta')\mathsf{e}^{\frac{\widehat{\mathsf{J}}}{2}(\eta\cdot\xi'-\eta'\cdot\xi)}}$$

We are then ready to quantize using a Weyl map. Given a function on phase space we associate to it a quaternionic operator:

$$\boxed{\widehat{\mathsf{W}}\,(f(x,p)) = \frac{1}{(2\pi)^3}\int \mathrm{d}x\mathrm{d}p\mathrm{d}\eta\mathrm{d}\xi \mathrm{e}^{-\widehat{\mathsf{J}}(\eta\cdot p+\xi\cdot x)}\widehat{\mathsf{T}}(\eta,\xi)\left(f_r(x,p)+\widehat{\mathsf{J}}f_i(x,p)\right)}$$

we can define a quantizer

$$\boxed{\Omega = \int \mathrm{d}\eta\mathrm{d}\xi \mathrm{e}^{-\widehat{\mathsf{J}}(\eta\cdot p+\xi\cdot x)}\widehat{\mathsf{T}}(\eta,\xi) = \int \mathrm{d}\eta\mathrm{d}\xi \mathrm{e}^{-\widehat{\mathsf{J}}(\eta\cdot p+\xi\cdot x)}\mathrm{e}^{\widehat{\mathsf{J}}(\eta\cdot\widehat{\mathsf{p}}+\xi\cdot\widehat{\mathsf{x}})}}$$

The inverse map need a dequantizer

$$\boxed{f(x,p) = \mathsf{W}^{-1}(\mathsf{f}) = \frac{1}{2}\left(\,\mathsf{Tr}\,\mathsf{tr}\,\mathsf{f}\,\Gamma - \mathsf{i}\,\mathsf{Tr}\,\mathsf{tr}\,\mathsf{f}\,\Gamma\right)}$$

And also in this case quantizer and dequantizer are the same

$$\boxed{\Omega = \Gamma}$$

I refrain from showing the calculation which is typical appendix material. One has to take into account all of the various phases which appear, and of course keep track of the fact that quaternions do not commute

Conclusions 1/2

Several of the features of Weyl quantization are preserved, for example real function go into self-adjoint operators

In fact the product closely mimic Weyl quantization for the usual case, the only central point is the substitution of the complex structure with $\boxed{\mathrm{j}(x)}$

This is quaternionic, position dependent, and spherically invariant. Notice moreover that I dealt with the monopole without introducing strings nor patches for the potential.

Conclusions 2/2

In some loose sense $\boxed{\mathrm{j}}$ reproduces what the Higgs field does for the 't Hooft-Polyakov monopole, and the analogy can be made more precise if one uses the usual representation of the quaternionic units in terms of Pauli matrices $\boxed{\mathrm{e}_i = \mathrm{i}\sigma_i}$

Apart from other uses in quantization, like having different Hamiltonian with the addition of other interactions, I am more curious on the connection with $\boxed{\mathrm{SU}(2)}$ which one can generalize in several directions, for example considering quantum deformations. . .