The Structure of Spacetime

&

Noncommutative Geometry

Pointers for a discussion

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Disclaimer: This is not a "regular" talk. Little of nothing of what I will describe has been contributed by me. It is rather a contribution to the discussion.

I will review some attempts that mathematicians and physicists are making to understand the structure of spacetime in the infinitesimally small

The tool used to this effect is Noncommutative Geometry

Let me start with a definition of Geometry

For which I can think of no higher authority than **Wikipedia**

Geometry (Greek $\gamma\varepsilon\omega\mu\varepsilon\tau\rho\iota\alpha$*; geo = earth, metria = measure) is a part of mathematics concerned with questions of size, shape, and relative position of figures and with properties of space.*

Geometry is at the hearth of several physical theories, including:

- **Classical Mechanics** The evolution of points in phase space

- **Special Relativity** The notion of event, a point in Minkowski space

- **Gravitation** The theory of a curved spacetime

And then we have:

- **String Theory** Probably a whole seminar would not suffice just to enumerate the numerous uses of geometry in string theory

In all these theories the geometry used is the mostly usual one, based on the concepts of points, lines etc.

When we move to quantum mechanics one important geometrical object, phase space, undergoes a drastic transformation

Heisenberg's uncertainty principle $\Delta x^i \Delta p^j \geq \frac{\hbar}{2}$ forces us to abandon the concept of point in phase space.

What is usually done is to consider x and p to be operators on an Hilbert space, and the uncertainty principle is a consequence of the relation $[x^i, p_j] = i\hbar\delta^i_j$

Geometry is still an useful tool, for example the Kähler geometry of the space of rays (Ashtekar-Schilling)

Is it legitimate to expect the usual geometry to hold to all scales?

There are several arguments which indicate physical reasons for which it should not be so

Just to mention one (Doplicher-Fredenhagen-Roberts) which evocates a similar reasoning for phase space:

In order to "measure" the position of an object, and hence the "point" in space, one has use a very small probe, and quantum mechanics forces us to have it very energetic, but on the other side general relativity tells us that if too much energy is concentrated in a region a black hole is formed.

The scale at which this happens is of the order of Planck's length

$$\ell_P = \sqrt{\frac{G\hbar}{c^3}} = 1.6\ 10^{-33}\ cm.$$

This is the region in which the theory to use is Quantum Gravity. Unfortunately a theory we do not yet have

In fact the two problems are related. A quantum gravity theory needs spacetime to be a different object from the one used in classical geometry

For example in loop quantum gravity 3-space is directly quantized and the geometry used there is certainly different from the classical one

Also in string theory spacetime undergoes changes

It is not anymore a given starting point but for example its dimensions emerge from the quantization of a conformal two-dimensional field theory

Interacting strings are described by the insertion of vertex operators on the worldsheet

At ultra high energy the structure of spacetime is again a (still somewhat mysterious) object in which ordinary spacetime has undergone strong transformations (M theory)

Noncommutative Geometry offers a systematic way to generalizes spaces, with its roots in quantum mechanics

First of all commutative geometry: *Hausdorff* topological spaces are on a one to one correspondence with *commutative* C^*-algebras (Gelfand-Najmark)

Given a space I can build a C^*-algebra: the algebra of continuous functions on it. Given an algebra I can reconstruct the space as the space of characters.

The generalization is "simply" done considering noncommutative C^*-algebras

For several years Connes and others have been writing a “dictionary" to go beyond mere topology and translate all geometric properties

For example the metric properties are encoded by a (generalized) Dirac Operator

Thus the emphasis for the study of a geometrical structure passes from the points to the fields

Historically the first appearance of Noncommutative Geometry in a physics paper was Witten's Open String Field Theory (1986)

String fields are seen as maps from a string configuration in space into complex numbers, with an enormous gauge symmetry (reparametrisation). After gauge fixing the role of differential is played by the BRS operator

Then in the 90's there was Connes' approach to the Standard Model

In this case the space is only “almost” noncommutative, in the sense that the noncommutative algebra describing space is Morita equivalent to a commutative algebra

In fact the algebra is a tensor product $\mathcal{A} = C(\mathbf{R}^4) \otimes \mathcal{A}_F$, with $\mathcal{A}_F$ a finite matrix algebra

The aim is not to predict the Lagrangian of standard model (taken as input) but to find a noncommutative geometry which describes the standard model

The model, especially in its last version (Chamseddine-Connes-Marcolli) has some predictive power (mass of the Higgs), but it is inherently classical, and once a Lagrangian is written, renormalization is performed in the usual way

Impulse to study noncommutative spaces came again from Strings Frohlich-Gawedski, Landi-FL-Szabo, Seiberg-Witten when it turned out that, in some limit, the vertex operators of a string theory show the behaviour given by noncommutative coordinates

In the spirit of what I said before one can threat a noncommuting space deforming the algebra of functions with a Grönewold-Moyal $\star$ product:

$$f \star g = f e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}} g$$

In this way we encode the noncommutativity of spacetime in the deformation of the algebra

Noncommutative Field Theory: Deformation of a commutative theory with the presence of a star product among the fields. For example

$$S = \int d^d x \partial_\mu \varphi \star \partial^\mu \varphi + m^2 \varphi \star \varphi + \frac{g^2}{4!} \varphi \star \varphi \star \varphi \star \varphi$$

For the Grönewold-Moyal product the $\star$ on the first two terms is redundant because $\int d^d x f \star g = \int d^d x f g$

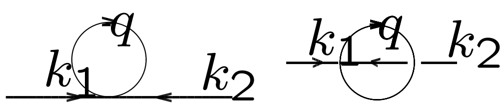
These theories are nonlocal, we have to abandon points because we have the analogous of Heisenberg's uncertainty principle

What physics comes out of these theories (and which cosmology?)

The free theory is unchanged because of the integral property. But the vertex gets a phase. For the example $\varphi^{\star 4}$:

$$V = (2\pi)^4 g \delta^4 \left(\sum_{a=1}^{4} k_a \right) \prod_{a<b} {}^{-\frac{i}{2}\theta^{\mu\nu} k_{a\mu} k_{b\nu}}$$

The vertex is not anymore invariant for exchange of the momenta (only for cyclic permutations), and causes a difference between planar and nonplanar diagrams



A consequence of this is Ultraviolet/Infrared Mixing Minwalla-Seiberg-Van Raamnsdong. The phenomenon for which some ultraviolet divergences disappear, just to reappear as infrared divergences

If we take seriously the fact that the world is described by this kind of noncommutative field theory which are the consequences? How do we measure $\theta^{\mu\nu}$, a quantity of the order of ℓ_P^2?

At this level, and as suggested by string theory, $\theta^{\mu\nu}$ is a background quantity, which selects two directions in space (analog of electric and magnetic fields). Their presence breaks Lorentz invariance and the noncommutativity will have left its imprinting in the early universe, and its consequences are thereafter frozen by inflation

Direct accelerator measurements are more difficult because the earth rotation washes up the effect. But one can look for otherwise forbidden processes

The problem is that the Moyal product is made for flat coordinates. The construction of associative deformed products is not simple (Kontsevich has won a field medal building them). One cannot simply substitute, say, the partial derivatives in the definition with covariant derivatives.

Nevertheless something has been done (Chu-Greene-Shiu, Brandemberger, FL-Mangano-Miele-Peloso).In our work we considered the field theory of a field which causes inflation to be deformed by a star product

We defined a curved star product to first order in $\theta = \theta^{12} = 1/\Lambda a^2$, a being the usual scale factor of the universe and Λ^{-1} a non-commutativity scale. Notice that the fact that θ is a tensor chooses a direction (12 in our case)

With this choice and a Moyal product defined with covariant derivatives nonassociativity is fourth order effect, and one can study the corrections to the inflaton action.

The corrections are such that the background evolves in the usual way, but the fluctuations change. Just for completeness let me flash the extra terms:

$$\delta S_V = -\frac{m^2}{16} \int d^4x\, a^3 \frac{1}{\Lambda^4} \left(\frac{\dot{a}}{a}\right)^2 \left(\partial_1\phi\partial^1\phi + \partial_2\phi\partial^2\phi\right), \tag{1}$$

$$\begin{aligned}
\delta S_K &= \frac{1}{32} \int d^4x \sqrt{-g}\, \Theta^{\mu\nu}\Theta^{\rho\sigma} \left(D_\rho D_\tau \phi\right) \left(\left[D_\mu, D_\nu\right] D_\sigma D^\tau \phi\right) \\
&= \frac{1}{16} \int d^4x\, a^3 \frac{1}{\Lambda^4} \left(\frac{\dot{a}}{a}\right)^2 \Big[\partial_m\partial_0\phi\partial^m\partial^0\phi + \partial_m\partial_i\phi\partial^m\partial^i\phi - \\
&\qquad - 2\frac{\dot{a}}{a}\partial^m\phi\partial_0\partial_m\phi + \left(\frac{\dot{a}}{a}\right)^2 \partial_m\phi\partial^m\phi\Big],
\end{aligned} \tag{2}$$

with $m = 1, 2$.

These terms have as a consequence a quadrupole contribution to the CMB (while gaussianity is preserved)

It is not easy however to distinguish predictions coming from these kind of theories from other breakings of Lorentz invariance

Since a noncommutative product is nonlocal, can this nonlocality have consequences in the early universe, give a different solution to the horizon problem for example?

But, given that we want to use noncommutativity of spacetime, are we sure we are using the right one? And what about the breaking of Lorentz symmetry in a fundamental theory? Notice that in the original Doplicher et al. paper $\boxed{\theta}$ is a central operator, and Lorentz is not broken

A deformation of spacetime may require a deformation of symmetries. **Quantum Groups and Hopf Algebras**

There is a Hopf algebra which is causing great interest:

θ**-Poincaré**

Consider the symmetry to be a twisted quantum symmetry (Wess and the Münich group: Aschieri, Blohmann, Dimitriević, Meyer, Schupp, Chaichian-Kulish-Nishijima-Tureanu, Oeckl, Majid, Drinfeld . . .)

Consider the usual action of the Lie algebra L of differential operators on the algebra $\mathcal{A}$ of functions with the usual commutative product

The usual product can be seen as a map from $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$

$$m_0(f \otimes g) = fg$$

pointwise multiplication

The Leibnitz rule imposes a coalgebra structure of the Lie algebra:

$$\ell(fg) = \ell(f)g + f\ell(g) = m_0(\Delta(L)(f \otimes g))$$

where ℓ is a generic first order differential operator

$$\Delta : L \to L \otimes L$$

$$\Delta(\ell) = \ell \otimes 1 + 1 \otimes \ell$$

The coproduct tells how to put together representations, and how an operator acts on two copies of the module.

Consider the Moyal product as follows

$$(f \star g)(x) = m_0[\mathcal{F}^{-1} f \otimes g] \equiv m_\theta[f \otimes g]$$

where

$$m_0(f \otimes g) = fg$$

is the ordinary product and

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_{x_\mu}\otimes\partial_{y_\nu}} = e^{-\frac{i}{2}\theta(\partial_{x_0}\otimes\partial_{y_1}-\partial_{x_1}\otimes\partial_{y_0})}$$

is called the twist.

The Noncommutative plane is obtained twisting the tensor product, and using the ordinary product.

We have to then revise the Leibnitz rule:

$$\partial_\mu(f \star g) = m_\theta \Delta_\theta(f \otimes g) = m_0 \Delta(\partial_m u)(\mathcal{F}^{-1}(f \otimes g))$$

where

$$\Delta_\theta = \mathcal{F} \Delta \mathcal{F}^{-1}$$

The algebra structure remains unchanged, what changes is the coalgebra structure, that is the way to "put together representations".

counit and antipode remain unchanged.

We have this deformed the coalgebra structure of the Poincaré Lie algebra. In particular:

The Lie algebra structure (commutators) is not changed. What changes is the coalgebra, at the level of the Lorentz group

$$\Delta_{\mathcal{F}}(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu$$

$$\Delta_{\mathcal{F}}(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta}\left((\eta_{\alpha\mu}P_\nu - \eta\alpha\nu P_\mu) \otimes P_\beta + P_\beta\left(\eta_{\beta\mu}P_\nu - \eta\beta\nu P_\mu\right)\right)$$

The fact that the algebra is the same means that we can still use the casimirs and the representations of the usual algebra, with thus concepts of mass, spin etc.

The twisted framework for noncommutative field theory is still under investigation, and is not free from controversies

We have changed the tensor product, and therefore one should twist all products in an appropriate way

Nevertheless there are already attempts at prediction, both in the gravitational framework, in the form of a deformed Einstein-Hilbert action (Wess et al.), or in the changes of statistics due to the twist.

But we are probably still lacking a “canonical” procedure to understand the twist

Another possibility could be κ-Minkowki. This is the homogenous space of the κ-Poincaré quantum group, and it is characterized by the commutation relations

$$[x_i, x_0] = i\lambda x_i, \quad [x_i, x_j] = 0$$

The commutation relations for κ-Poincaré are:

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[M_i, P_j] &= i\epsilon_{ijk} P_k \\
[M_i, P_0] &= 0 \\
[N_i, P_j] &= -i\delta_{ij}\left(\frac{1}{2\lambda}(1 - e^{2\lambda P_0}) + \frac{\lambda}{2} P^2\right) + i\lambda P_i P_j \\
[N_i, P_0] &= iP_i \\
[M_i, M_j] &= i\epsilon_{ijk} M_k \\
[M_i, N_j] &= i\epsilon_{ijk} N_k \\
[N_i, N_j] &= -i\epsilon_{ijk} M_k
\end{aligned}
$$

All these commutation relations become the standard ones for $\lambda \to 0$. The bicrossproduct basis is peculiar as κ-Poincaré acts *covariantly* on a space that is necessarily deformed and noncommutative. This is a consequence of the non cocommutativity of the coproduct which, always in the bicrossproduct basis, reads:

$$
\begin{aligned}
\Delta P_0 &= P_0 \otimes 1 + 1 \otimes P_0 \\
\Delta M_i &= M_i \otimes 1 + 1 \otimes M_i \\
\Delta P_i &= P_i \otimes 1 + e^{\lambda P_0} \otimes P_i \\
\Delta N_i &= N_i \otimes 1 + e^{+\lambda P_0} \otimes N_i + \lambda \varepsilon_{ijk} P_j \otimes M_k
\end{aligned}
$$

The Casimir of this quantum group provide a deformation of the Energy-Momentum dispersion relation and this could be used to explain γ-ray bursts (Amelino-Camelia). The problem is that, being the commutation relations nonlinear, nonlinear changes of coordinates are allowed, and therefore these dispersion relations become basis-dependent.

Is it possible to draw conclusions?

My personal conclusion is that at the Planck scale there should be a noncommutative structure, but we may not have gotten yet the right one

Noncommutative geometry is actually more a tool than a theory, and it should probably complement a more general theory (probably some version of strings or loop quantum gravity

Fortunately we can expect some input from experiments and observations: LHC, Planck, cosmic rays