Noncommutative Field Theory with the

Moyal and Wick-Voros products:

&

Twist and s-Matrix

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The starting point for a large part of what is now called non-commutative geometry is the commutator

$$\boxed{x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}}$$

This is implemented via the Moyal product

$$\boxed{f(\vec{x}) \star_{\mathsf{M}} g(\vec{x}) = f(\vec{x}) \mathrm{e}^{\frac{i}{2}\theta^{ij}\overleftarrow{\partial_i}\overrightarrow{\partial_j}} g(\vec{x})}$$

This product is not the only one which gives the above commutation relation

In the following I will be in $\boxed{2+1}$ dimensions with only spatial noncommutativity $\boxed{\theta^{ij} = \theta \varepsilon ij}$

Introduce the Wick-Voros:

$$\boxed{z_{\pm} = \frac{x^1 \pm i x^2}{\sqrt{2}}}$$

$$\boxed{\partial_{\pm} = \frac{\partial}{\partial z_{\pm}} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x^1} \mp i \frac{\partial}{\partial x^2} \right)}$$

$$\boxed{f \star_{\mathrm{V}} g = \sum_n \left(\frac{\theta^n}{n!} \right) \partial_+^n f \partial_-^n g = f \mathrm{e}^{\theta \overleftarrow{\partial_+} \overrightarrow{\partial_-}} g}$$

This product is also called normal ordered product because it comes from a normal ordering quantization map which associates harmonic oscillator creation and annihilation operators to $\boxed{z_{\pm}}$

We studied a $\boxed{\varphi^{\star 4}}$ field theory with the Wick-Voros product and compared it with the one with the Moyal product

Note a crucial difference between the Moyal and Wick-Voros product:

$$\boxed{\int \mathrm{d}^2 z f \star_{\mathrm{V}} g = \int \mathrm{d}^2 z\, g \star_{\mathrm{V}} f \neq \int \mathrm{d}^2 z\, f g = \int \mathrm{d}^2 z f \star_{\mathrm{M}} g}$$

This means that also the free theory will be different from the undeformed case. The two free propagators are

$$\boxed{G^{(2)}_{0_M}(p) = \frac{1}{p^2 - m^2}}$$

$$\boxed{G^{(2)}_{0_V}(p) = \frac{\mathrm{e}^{-\frac{\theta}{2}|\vec{p}|^2}}{p^2 - m^2}}$$

Similarly it is possible to calculate vertex and tree level four points Green's function:

$$G_0^{(4)} = \frac{\mathrm{e}^{\sum_{a\le b} k_a \bullet k_b}}{\prod_{a=1}^{4}\left(k_a^2 - m^2\right)}\delta\left(\sum_{a=1}^{4} k_a\right)$$

where

$$k_a \bullet k_b = \begin{cases} -\frac{i}{2}\theta^{ij} k_{a_i} k_{b_j} & \text{Moyal} \\ \\ -\theta k_{a-} k_{b+} = -\frac{1}{2}\left(\theta k_{a_i} {k_b}^i + i\theta^{ij} k_{a_i} k_{b_j}\right) & \text{Wick-Voros} \end{cases}$$

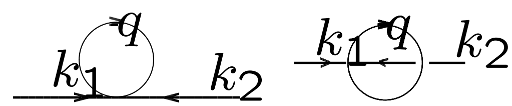
The two products however are equivalent in the following sense: there is an invertible map $\boxed{T = \mathrm{e}^{\frac{\theta}{4}\nabla^2}}$ with the property

$$\boxed{T(f \star_{\mathsf{M}} g) = T(f) \star_{\mathsf{V}} T(g)}$$

Therefore the two deformations define the same deformed algebra, and one could expect them to give the same "physics"

For example one could reduce the whole theory to a matrix model and quantize it with a path integral

Calculate the four point amplitude at one loop for the planar and nonplanar diagrams:



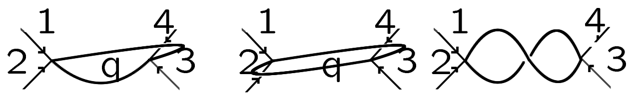
$$G_{\mathrm{P}}^{(2)} = g\frac{1}{2}\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\mathrm{e}^{p\bullet p}}{(p^2 - m^2)^2 (q^2 - m^2)}$$

$$G_{\mathrm{N}P}^{(2)} = g\frac{1}{2}\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\mathrm{e}^{p\bullet p + p\bullet q - q\bullet p}}{(p^2 - m^2)^2 (q^2 - m^2)}$$

Notice that $\boxed{p \bullet q - q \bullet p = i\vec{p} \wedge \vec{q}}$ in both cases, but the two Green's functions are not the same because of the $\boxed{p \bullet p}$ term.

The ultraviolet/infrared mixing appears in the same way in both cases

The four points one-loop Green's functions correspond to the planar and the three nonplanar cases



For the planar case the result is

$$\boxed{G_{\mathsf{P}}^{(4)} = (2\pi)^6 g^2 \int \mathrm{d}q \frac{\mathrm{e}^{\sum_{a\leq b} k_a \bullet k_b}\ \delta\left(\sum_{a=1}^4 k_a\right)}{(q^2-m^2)((k_1+k_2-q)^2-m^2)\prod_{a=1}^4 (k_a^2-m^2)}}$$

While for the undeformed case there is an extra term

$$G^{(4)}_{\mathrm{NP_a}} = (2\pi)^6 \int \mathrm{d}q \frac{\mathrm{e}^{\sum_{a\le b} k_a \bullet k_b + E_a} \ \delta\left(\sum_{a=1}^4 k_a\right)}{(q^2 - m^2)((k_1 + k_2 - q)^2 - m^2) \prod_{a=1}^4 (k_a^2 - m^2)}$$

with

$$E_1 = q \bullet k_1 - k_1 \bullet q = i\vec{q} \wedge \vec{k}_1$$

$$E_2 = k_2 \bullet q - q \bullet k_2 + k_3 \bullet q - q \bullet k_3 = i(\vec{k}_2 \wedge \vec{q} + \vec{k}_3 \wedge \vec{q})$$

$$E_3 = k_1 \bullet q - q \bullet k_1 + k_2 \bullet q - q \bullet k_2 = i(\vec{k}_1 \wedge \vec{q} + \vec{k}_2 \wedge \vec{q})$$

Notice that these extra terms are the same in both cases

The phases in the nonplanar diagrams are the same in the two cases, but the exponent $\boxed{\mathrm{e}^{\sum_{a\leq b} k_a \bullet k_b}}$ is different

Again the ultraviolet/infrared is the same for Moyal and Wick-Voros

For the Moyal case the exponent is just a phase, but for Wick-Voros there is also a real part $\boxed{\theta k_a \cdot k_b}$, something like a momentum dependent coupling constant

Curiously this term is large or small depending on the sign of theta

At any rate the Green's function in the two cases are different, contrary to our expectation that the two theories should describe the same "physics"

However Green's functions are not directly measurable, S-matrices are

And a discussion about S-matrix immediately calls for one on Poincaré symmetry. In the case of both theories the symmetry is a deformed symmetry, a quantum group (Hopf algebra) obtained by a Drinfeld twist

Consider the twist, an operator which acts on the tensor product of function, which in momentum spaces is, for the two cases:

$$\boxed{\mathcal{F}_{\star\mathrm{M}}^{-1} = \mathrm{e}^{\frac{i}{2}\theta^{ij}\partial_i\otimes\partial_j}}$$

$$\boxed{\mathcal{F}_{\star\mathrm{V}}^{-1} = \mathrm{e}^{\theta\partial_+\otimes\partial_-}}$$

We take the point of view (Aschieri et. al.) that we have to twist with the above operator, in its proper representation, all products encountered

Given a generic product from the product of two spaces into a third

$$\boxed{\mu : \; X \times Y \longrightarrow Z}$$

we associate a deformed product

$$\boxed{\mu_\star : \; \mathcal{F}^{-1}(X \times Y) \longrightarrow Z}$$

In particular when $\boxed{X = Y = Z = C(M)}$, the algebra of functions on a manifold, the usual pointwise product is deformed in the appropriate $\boxed{\star}$ product

We set on calculating the S-matrix at one loop for the four points amplitude twisting all products we encountered

Hoping we catch them all!

In this way we are actually playing Moyal and Wick-Voros one against the other. We wish to see that, if we properly define a twisted theory, we can obtain the same results.

Step 0: define the free asymptotic state

$$\boxed{|k\rangle = a_k^\dagger |0\rangle}$$

where

$$\boxed{a_k^\dagger = -\frac{i}{\sqrt{(2\pi)^2 2\omega_k}} \int d^2x \mathrm{e}^{-ik\cdot x} \overleftrightarrow{\partial_0} \varphi_{in}(x)}$$

$\boxed{a}$ and $\boxed{a^\dagger}$ are functionals of the fields, and therefore, using the twist, we can define a deformed product for them:

$$\boxed{a(k) \star a(k') = \tilde{\mathcal{F}}^{-1} a(k)a(k') = \mathrm{e}^{k\bullet k'} a(k)a(k')}$$

etc.

Note: Whenever we use $\boxed{\star}$ without specification it means that the expression is valid in both cases

Step 1 Two particles state:

$$\boxed{\tilde{\mathcal{F}}_{\star\mathsf{M}}^{-1}\,|k_a\rangle\otimes|k_b\rangle = |k_a,k_b\rangle_{\star\mathsf{M}} = \mathrm{e}^{-\frac{i}{2}\theta^{ij}k_{a_i}\otimes k_{b_j}}\,|k_a\rangle\otimes|k_b\rangle}$$

$$\boxed{\tilde{\mathcal{F}}_{\star\mathsf{V}}^{-1}\,|k_a\rangle\otimes|k_b\rangle = |k_a,k_b\rangle_{\star\mathsf{V}} = \mathrm{e}^{\theta k_{a_-}\otimes k_{b_+}}\,|k_a\rangle\otimes|k_b\rangle}$$

Which can be expressed in an unified way as:

$$\boxed{|k_a,k_b\rangle_\star = a^\dagger_{k_a}\star a^\dagger_{k_b}\,|0\rangle}$$

Step 2 Inner product among one particle states

Also the inner product should be twisted giving

$$\boxed{\left\langle \cdot \overset{\star}{\Big|} \cdot \right\rangle : |k\rangle \otimes \left|k'\right\rangle \longrightarrow \langle \cdot | \cdot \rangle \circ \mathcal{F}^{-1} : |k\rangle \otimes \left|k'\right\rangle = \tilde{\mathcal{F}}^{-1}(k,k') \left\langle k | k' \right\rangle}$$

$$\boxed{= \langle 0| \, a_k \star a^{\dagger}_{k'} \, |0\rangle}$$

Step 3 Inner product among two-particle states

To act on two particle states we have to put together representations. For Hopf algebras this is done via the coproduct, a structure of Hopf algebras:

$$\boxed{\Delta_\star(u)(f \otimes g) = (1 \otimes u + \mathcal{R}^{-1}(u \otimes 1))(f \otimes g)}$$

In this particular case, because translations commute the twist of the coproduct is unimportant $\boxed{\Delta_0 = \Delta_\star}$

We obtain

$$\boxed{\left\langle k_1 k_2 \overset{\star}{\Big|} k_3 k_4 \right\rangle = \langle \cdot | \cdot \rangle \circ \Delta_\star(\mathcal{F}^{-1})(|k_1 k_2\rangle \otimes |k_3 k_4\rangle)}$$

where by $\boxed{\Delta_\star(\mathcal{F}^{-1})}$ is defined by $\boxed{\Delta(\partial \otimes \partial) = \Delta(\partial) \otimes \Delta(\partial)}$

$$\boxed{\left\langle k_1, k_2 \overset{\star M}{\Big|} k_3, k_4 \right\rangle = \mathrm{e}^{\frac{i}{2}\theta^{ij}(k_{1_i} + k_{2_i})(k_{3_j} + k_{4_j})} \langle k_1, k_2 | \, k_3, k_4 \rangle}$$

$$\boxed{\left\langle k_1, k_2 \overset{\star V}{\Big|} k_3, k_4 \right\rangle = \mathrm{e}^{\theta(k_{1-} + k_{2-})(k_{3+} + k_{4-})} \langle k_1, k_2 | \, k_3, k_4 \rangle}$$

The expression for the inner product can be concisely written as:

$$\boxed{{}_{\star}\!\left\langle k_1, k_2 \,\middle|^{\star}\, k_3, k_4 \right\rangle_{\star} = \langle 0 |\, a_{k_1} \star a_{k_2} \star a^{\dagger}_{k_3} \star a^{\dagger}_{k_4} \,| 0 \rangle}$$

$$\boxed{= \mathrm{e}^{-\sum_{a<b} k_a \bullet k_b} \langle k_1, k_2 | \, k_3, k_4 \rangle}$$

Which is a rather "natural" expression

We are now ready to calculate the S-matrix two particles elastic scattering (to one loop)

$$\boxed{S_{fi} = {}_{in\star}\left\langle f \overset{\star}{\Big|} i \right\rangle_{\star out} = {}_{in\star}\left\langle f \overset{\star}{\Big|} S \overset{\star}{\Big|} i \right\rangle_{\star in}}$$

The one-particle asymptotic state is

$$\boxed{|k\rangle_{in} = N_\star(k) a_k^\dagger \, |0\rangle_{in} = -N_\star(k) \frac{i}{\sqrt{(2\pi)^2 2\omega_k}} \int d^2x \mathrm{e}^{-ik\cdot x} \overleftrightarrow{\partial_0} \, \varphi_{in}(x) \, |0\rangle_{in}}$$

with $\boxed{N_\star(k)}$ a normalization factor to be determined for the Moyal and Voros cases

We assume

$$\boxed{\lim_{x^0 \to \pm\infty} \langle f|\varphi(x)|i\rangle = Z^{1/2} \langle f|\varphi_{\substack{out \\ in}}(x)|i\rangle}$$

$\boxed{Z}$ a renormalization factor

We obtain:

$$\boxed{S_{fi} = \quad {}_{in\star}\left\langle k_1 k_2 \;\overset{\star}{\Big|}\; k_3 k_4 \right\rangle_{out\star} \quad = \text{disconnected}}$$

$$\boxed{+\bar{N}_\star(k_1)\bar{N}_\star(k_2)N_\star(k_3)N_\star(k_4)(iZ^{-1/2})^2 \mathrm{e}^{-\sum_{a<b} k_a \bullet k_b}}$$

$$\boxed{\times \int \frac{\prod_a d^2x^a}{\sqrt{(2\pi)^2 2\omega_{k_a}}} \mathrm{e}^{-ik_a x^a}(\partial_\mu^2 + m^2)_a G(x^1, x^2, x^3, x^4)}$$

where $\boxed{G(x^1, x^2, x^3, x^4)}$ is the four-point Green's function.

to fix the normalization of the asymptotic states compute the scattering amplitude for one particle going into one particle, at zeroth order. Up to the undeformed normalization factors $\boxed{N(p_a)}$, this has to give a delta function:

$$\boxed{N^*(k)N(p)\delta^2(k-p) = N_\star^*(k)N_\star(p) \quad {}_{in\star}\left\langle k \overset{\star}{\Big|} p \right\rangle_{out\star}}$$

$$\boxed{= N_\star^*(k)N_\star(p)\mathrm{e}^{-k\bullet p} \ {}_{in}\langle k|\, p\rangle_{out} = N_\star^*(k)N_\star(p)\mathrm{e}^{-k\bullet p}\delta^2(k-p)}$$

from which we obtain

$$\boxed{N_{\star_\mathrm{M}}(p) = N(p)}$$

$$\boxed{N_{\star_\mathrm{V}}(p) = \mathrm{e}^{-\frac{\theta}{4}|\vec{p}|^2} N(p)}$$

For the planar case we have

$$\boxed{S_{fi_{\star P}}(k_1,..,k_4) = \bar{N}(k_1)\bar{N}(k_2)N(k_3)N(k_4)\Pi_a \mathrm{e}^{\frac{\theta}{4}|\vec{k}_a|^2}}$$

$$\boxed{\mathrm{e}^{-\sum_{a<b} k_a \bullet k_b} \int \Pi_a \frac{d^2x^a}{\sqrt{(2\pi)^2 2\omega_{k_a}}} \mathrm{e}^{-ik_a x^a} \int \Pi_a \frac{d^2 p_a}{\sqrt{(2\pi)^2 2\omega_{p_a}}} \mathrm{e}^{ip_a x^a}(-p_a^2 + m^2)}$$

$$\boxed{\int \mathrm{d}q \frac{\mathrm{e}^{\sum_{a\le b} p_a \bullet p_b}}{(q^2-m^2)((p_1+k_2-q)^2-m^2)\prod_{a=1}^4 (p_a^2 - m^2)} \delta\left(\sum_{a=1}^4 p_a\right)}$$

The propagators of the external legs and the exponent cancel in the $\boxed{x}$ integration so that we are left with the usual undeformed expression

$$\boxed{S_{fi_{\star P}}(k_1,..,k_4) = S_{fi}(k_1,..,k_4)}$$

The nonplanar case is different form the undeformed case:

$$\boxed{S_{fi_{\star NP}}(k_1,..,k_4) = \bar{N}(k_1)\bar{N}(k_2)N(k_3)N(k_4)\Pi_a e^{\frac{\theta}{4}|\vec{k}_a|^2}}$$

$$\boxed{e^{-\sum_{a<b} k_a \bullet k_b} \int \Pi_a \frac{d^2x^a}{\sqrt{(2\pi)^2 2\omega_{k_a}}} e^{-ik_a x^a} \int \Pi_a \frac{d^2p_a}{\sqrt{(2\pi)^2 2\omega_{p_a}}} e^{ip_a x^a}(-p_a^2 + m^2)}$$

$$\boxed{\int dq \frac{e^{\sum_{a\leq b} p_a \bullet p_b + E_a}}{(q^2 - m^2)((p_1 + k_2 - q)^2 - m^2)\prod_{a=1}^4 (p_a^2 - m^2)} \delta\left(\sum_{a=1}^4 p_a\right)}$$

After integrating the previous cancellations still hold, but but we are left with the exponential of $\boxed{E_a}$ which doesn't simplify.

It is an imaginary phase, and it has the same expression in the Moyal and Wick-Voros case. It depends on the $\boxed{q}$ variable, therefore it gets integrated and modifies the IR and UV behaviour of the loop: this is the correction responsible for the UV/IR mixing

Conclusions

We have investigated two different but equivalent noncommutative $\boxed{\star}$ product. With the prejudice that they should give the same “physical predictions"

We have found that they give different Green’s functions and free propagators

We have also found that if the calculations of the S-matrix are made in a controlled way, deforming all products, then the result is indeed the same for the two noncommutative theories, but it is different from the undeformed case