Noncommutative Field Theory with the Moyal and Wick-Voros products: & Twist and s-Matrix

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The starting point for a large part of what is now called non-commutative geometry is the commutator

$$\boxed{x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}}$$

This is usually implemented via the Moyal product often written in the asymptotic form:

$$\boxed{f(x) \star_{\mathrm{M}} g(x) = f(x) \mathrm{e}^{\frac{i}{2}\theta^{ij}\overleftarrow{\partial_i}\overrightarrow{\partial_j}} g(x)}$$

It is a noncommutative, associative product introduced originally in quantum mechanics. It comes from a Weyl map between functions and operators (which enables better and more solid integral definitions of the product).

The commutation relation has been introduced in the spacetime context by Doplicher, Fredenhagen and Roberts. It became famous when it emerged in the product between vertex operators of strings.

The Moyal product is not the only product which gives the above commutation relation

In this seminar I will discuss an alternative product, and compare the field theories built with it and with the Moyal product

In reality I have also an hidden agenda: **symmetries**

The issue of symmetries of physical theories is an important one, and the presence of $\boxed{\theta^{ij}}$ breaks Lorentz invariance

This invariance can be however recuperated in a quantum version, as a noncocommutative Hopf algebra

But the proper way to implement this quantum symmetry and its "physical" consequences is not completely immediate. We have experience with the usual, commutative theory and its usual, cocommutative symmetries.

The construction of a theory in this different context should start form a recognition of the physical objects, and their operational meaning. Then the theory is developed, often along more than one direction, confrontation with experiment indicating the correct way to proceed in ambiguous cases

The problem is that we do not have the help of experiment, and therefore the resolution of some ambiguities is difficult to establish.

In this respect the comparison of the theories may (and will) help in the understanding of a canonical procedure for the implementation of this kind of symmetries

In the following I will be in $\boxed{2+1}$ dimensions with only spatial noncommutativity $\boxed{\theta^{ij} = \theta\varepsilon^{ij}}$

Introduce the Wick-Voros product:

$$\boxed{z_\pm = \frac{x^1 \pm i x^2}{\sqrt{2}}}$$

$$\boxed{\partial_\pm = \frac{\partial}{\partial z_\pm} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^1} \mp i \frac{\partial}{\partial x^2}\right)}$$

$$\boxed{f \star_V g = \sum_n \left(\frac{\theta^n}{n!}\right) \partial_+^n f \partial_-^n g = f \mathrm{e}^{\theta \overleftarrow{\partial_+} \overrightarrow{\partial_-}} g}$$

This product still gives the usual commutation relation

$$\boxed{x^i \star_{\mathsf{V}} x^j - x^j \star_{\mathsf{V}} x^i = i\theta^{ij}}$$

it comes from a variation of the Weyl map:

$$\boxed{\hat{\Omega}_V(f) = \frac{1}{2\pi} \int \mathrm{d}^2\eta \tilde{f}(\eta, \bar{\eta}) \mathrm{e}^{\theta\eta a^\dagger} \mathrm{e}^{-\theta\bar{\eta}a}}$$

so that we can define it as

$$\boxed{f \star_{\mathsf{V}} g = \Omega_V^{-1} \left(\hat{\Omega}_V(f) \hat{\Omega}_V(g) \right)}$$

The product is also called normal ordered product because it associates harmonic oscillator creation and annihilation operators to $\boxed{z_\pm}$ with $\boxed{\hat{\Omega}(f(z_+, z_-) =: f(a^\dagger, a) :}$

We studied a $\boxed{\varphi^{\star 4}}$ field theory with the Wick-Voros product and compared it with the one with the Moyal product

$$\boxed{S_0 = \int \mathrm{d}t L_0 = \int \mathrm{d}t\mathrm{d}^2 z\, \mathcal{L}_0 = \int \mathrm{d}t\mathrm{d}^2 z\, \frac{1}{2}\left(\partial_\mu\varphi \star \partial_\mu\varphi - m^2\varphi \star \varphi\right)}$$

$$\boxed{S = S_0 + \frac{g}{4!}\int \mathrm{d}t\mathrm{d}^2 z\, \varphi \star \varphi \star \varphi \star \varphi}$$

Where $\boxed{\star}$ is either product

Note a crucial difference between the Moyal and Wick-Voros products:

$$\boxed{\int \mathrm{d}^2 z f \star_{\mathrm{V}} g = \int \mathrm{d}^2 z\, g \star_{\mathrm{V}} f \neq \int \mathrm{d}^2 z\, f g = \int \mathrm{d}^2 z f \star_{\mathrm{M}} g}$$

This means that also the free theory will be different from the undeformed case. The two free propagators are

$$\boxed{G^{(2)}_{0_M}(p) = \frac{1}{p^2 - m^2}}$$

$$\boxed{G^{(2)}_{0_V}(p) = \frac{\mathrm{e}^{-\frac{\theta}{2}|\vec{p}|^2}}{p^2 - m^2}}$$

Similarly it is possible to calculate vertex and tree level four points Green's function, generalising Filk's work:

$$G_0^{(4)} = -ig(2\pi)^3 \frac{\mathrm{e}^{\sum_{a\le b} k_a \bullet k_b}}{\prod_{a=1}^4 (k_a^2 - m^2)} \delta\left(\sum_{a=1}^4 k_a\right)$$

where

$$k_a \bullet k_b = \begin{cases} -\frac{i}{2}\theta^{ij} k_{a_i} k_{b_j} & \text{Moyal} \\ \\ -\theta k_{a-} k_{b+} = -\frac{1}{2}\left(\theta k_{a_i} {k_b}^i + i\theta^{ij} k_{a_i} k_{b_j}\right) & \text{Wick-Voros} \end{cases}$$

The Moyal case has just a phase, the Wick one has also a real part in the exponent

The two products however are equivalent in the following sense:

there is an invertible map $\boxed{T = \mathrm{e}^{\frac{\theta}{4}\nabla^2}}$ with the property:

$$\boxed{T(f \star_{\mathsf{M}} g) = T(f) \star_{\mathsf{V}} T(g)}$$

In fact they correspond to two equivalent "quantization" schemes

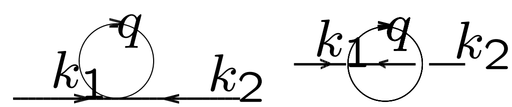
I can take two points of view;

I know nothing of the structure of spacetime, noncommutative geometry and all that. I am in the presence of two different actions, slightly unusual because they have an infinite number of derivatives, but under control.

Quantize spacetime, i.e. map functions to operators. Then to (second) quantize the action, I could forget about the underlying space, and perform my calculations at the operator level with a path integral. Nevertheless it is convenient to map the operators to functions with a star product, and quantize its fields in a conventional way. In this way I keep separated the quantization of spacetime from the quantization of fields.

In this view I expect the same "physics" with the equivalent products.

Calculate the four point amplitude at one loop for the planar and nonplanar diagrams:



$$\boxed{G_{\mathsf{P}}^{(2)} = g\frac{1}{2}\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\mathrm{e}^{p \bullet p}}{(p^2 - m^2)^2 (q^2 - m^2)}}$$

$$\boxed{G_{\mathsf{N}P}^{(2)} = g\frac{1}{2}\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\mathrm{e}^{p \bullet p + p \bullet q - q \bullet p}}{(p^2 - m^2)^2 (q^2 - m^2)}}$$

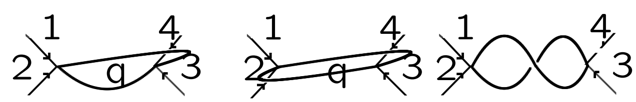
Notice that $\boxed{p \bullet p = 0}$ for the Moyal case, or $\boxed{-\theta |\vec{p}|^2}$ for Wick-Voros, while $\boxed{p \bullet q - q \bullet p = i p_i \theta^{ij} q_j}$ in both cases, but the two Green's functions are not the same because of the $\boxed{p \bullet p}$ term.

The phase $\boxed{ip_i\theta^{ij}q_j}$ is the one responsible for the ultraviolet/infrared mixing because the nonpalanar diagram is convergent at high momenta, but it develops infrared divergences.

Since the depedence on the internal momentum $\boxed{q}$ is the same in both cases, the ultraviolet/infrared mixing is unchanged

An heuristic way to see the mixing is as a consequence of a space uncertainty principle, which depends only on the commutation relations and is unchanged in this case.

The four points one-loop Green's functions correspond to the planar and the three nonplanar cases



For the planar case the result is

$$\boxed{G_{\mathsf{P}}^{(4)} = (2\pi)^6 g^2 \int \mathrm{d}q \frac{\mathrm{e}^{\sum_{a\leq b} k_a \bullet k_b}\ \delta\left(\sum_{a=1}^{4} k_a\right)}{(q^2 - m^2)((k_1 + k_2 - q)^2 - m^2)\prod_{a=1}^{4}(k_a^2 - m^2)}}$$

While for the nonplanar case there is an extra term

$$\boxed{G^{(4)}_{\mathsf{NP_a}} = (2\pi)^6 \int \mathrm{d}q \frac{\mathrm{e}^{\sum_{a\le b} k_a \bullet k_b + E_a}\ \delta\left(\sum_{a=1}^4 k_a\right)}{(q^2-m^2)((k_1+k_2-q)^2-m^2)\prod_{a=1}^4 (k_a^2-m^2)}}$$

with

$$\boxed{E_1 = q \bullet k_1 - k_1 \bullet q = i\vec{q} \wedge \vec{k}_1}$$

$$\boxed{E_2 = k_2 \bullet q - q \bullet k_2 + k_3 \bullet q - q \bullet k_3 = i(\vec{k}_2 \wedge \vec{q} + \vec{k}_3 \wedge \vec{q})}$$

$$\boxed{E_3 = k_1 \bullet q - q \bullet k_1 + k_2 \bullet q - q \bullet k_2 = i(\vec{k}_1 \wedge \vec{q} + \vec{k}_2 \wedge \vec{q})}$$

Notice that these extra terms are the same in both cases

The phases in the nonplanar diagrams are the same in the two cases, but the exponent $\boxed{e^{\sum_{a\leq b} k_a \bullet k_b}}$ is different

Again the ultraviolet/infrared is the same for Moyal and Wick-Voros

For the Moyal case the exponent is just a phase, but for Wick-Voros there is also a real part $\boxed{\theta k_a \cdot k_b}$, something like a momentum dependent coupling constant

Curiously this term is large or small depending on the sign of theta

At any rate the Green's function in the two cases are different, contrary to our expectation that the two theories should describe the same "physics"

However Green's functions are not directly measurable, so let us go one step up and discuss the S-matrix.

The issue of the determination of an S-matrix in noncommutative field theory is not a settled one, there are several issues yet to be clarified, mainly in relation to the asymptotic states. And we have no experiments to guide us...

The scope of the following discussion is to use the two products one against the other to find a procedure which gives the same results for both. This is not going to solve all problems that the theory might have. We only hope to clarify the role of deformed symmetries

We have to discuss Poincaré symmetry. In the case of both theories the symmetry is a deformed symmetry, a quantum group (Hopf algebra) obtained by a Drinfeld twist

I will be sketchy in the description of twisted theories which have been introduced in this context by the Munich (Aschieri, Blohmann, Dimitrijevic, Meyer, Schupp and Wess) and Helsinki (Chaichian, Kulish, Nishijima and Tureanu) groups, and discussed by other speakers at this meeting

Consider the twist, an operator which acts on the tensor product of function. For the two cases we have:

$$\boxed{\mathcal{F}_{\star\mathrm{M}}^{-1} = \mathrm{e}^{\frac{i}{2}\theta^{ij}\partial_i\otimes\partial_j}}$$

$$\boxed{\mathcal{F}_{\star\mathrm{V}}^{-1} = \mathrm{e}^{\theta\partial_+\otimes\partial_-}}$$

We take the point of view (Aschieri) that we have to twist with the above operator, in its proper representation, all products encountered

Given a generic product from the product of two spaces into a third

$$\boxed{\mu : \; X \times Y \longrightarrow Z}$$

we associate a deformed product

$$\boxed{\mu_\star : \; \mu \circ \mathcal{F}^{-1}(X \otimes Y) \longrightarrow Z}$$

In particular when $\boxed{X = Y = Z = C(M)}$, the algebra of functions on a manifold, the usual pointwise product is deformed in the appropriate $\boxed{\star}$ product

We set on calculating the S-matrix at one loop for the four points amplitude twisting all products we encountered

Step 0: define the free asymptotic state

$$\boxed{|k\rangle = a_k^\dagger |0\rangle}$$

where

$$\boxed{a_k^\dagger = -\frac{i}{\sqrt{(2\pi)^2 2\omega_k}} \int d^2x \mathrm{e}^{-ik\cdot x} \overleftrightarrow{\partial_0} \varphi_{in}(x)}$$

$\boxed{a}$ and $\boxed{a^\dagger}$ are functionals of the fields, and therefore, using the twist, we can define a deformed product for them:

$$\boxed{a(k) \star a(k') = \tilde{\mathcal{F}}^{-1} a(k)a(k') = \mathrm{e}^{k\bullet k'} a(k)a(k')}$$

etc.

Step 1 Two particles state:

$$\boxed{\tilde{\mathcal{F}}_{\star\mathsf{M}}^{-1}\,|k_a\rangle \otimes |k_b\rangle = |k_a, k_b\rangle_{\star\mathsf{M}} = \mathsf{e}^{-\frac{i}{2}\theta^{ij}k_{a_i}\otimes k_{b_j}}\,|k_a\rangle \otimes |k_b\rangle}$$

$$\boxed{\tilde{\mathcal{F}}_{\star\mathsf{V}}^{-1}\,|k_a\rangle \otimes |k_b\rangle = |k_a, k_b\rangle_{\star\mathsf{V}} = \mathsf{e}^{\theta k_{a_-}\otimes k_{b_+}}\,|k_a\rangle \otimes |k_b\rangle}$$

Which can be expressed in an unified way as:

$$\boxed{|k_a, k_b\rangle_\star = a^\dagger_{k_a} \star a^\dagger_{k_b}\,|0\rangle}$$

Step 2 Inner product among one particle states

Also the inner product should be twisted giving

$$\boxed{\left\langle \cdot \overset{\star}{\middle|} \cdot \right\rangle : |k\rangle \otimes \left|k'\right\rangle \longrightarrow \langle \cdot | \cdot \rangle \circ \mathcal{F}^{-1} : |k\rangle \otimes \left|k'\right\rangle = \tilde{\mathcal{F}}^{-1}(k, k') \langle k | k' \rangle}$$

$$\boxed{= \langle 0 | a_k \star a^{\dagger}_{k'} |0\rangle}$$

Step 3 Inner product among two-particle states

To act on two particle states we have to put together representations. For Hopf algebras this is done via the coproduct, a structure of Hopf algebras:

$$\boxed{\Delta_\star(u)(f \otimes g) = (1 \otimes u + \mathcal{R}^{-1}(u \otimes 1))(f \otimes g)}$$

In this particular case, because translations commute the twist of the coproduct is unimportant $\boxed{\Delta_0 = \Delta_\star}$

We obtain

$$\boxed{\left\langle k_1 k_2 \,\middle|^{\star}\, k_3 k_4 \right\rangle = \langle \cdot | \cdot \rangle \circ \Delta_\star(\mathcal{F}^{-1})(\, |k_1 k_2\rangle \otimes |k_3 k_4\rangle)}$$

where by $\boxed{\Delta_\star(\mathcal{F}^{-1})}$ is defined by $\boxed{\Delta(\partial \otimes \partial) = \Delta(\partial) \otimes \Delta(\partial)}$

$$\boxed{\left\langle k_1, k_2 \,\middle|^{\star \mathrm{M}}\, k_3, k_4 \right\rangle = \mathrm{e}^{\frac{i}{2}\theta^{ij}(k_{1_i}+k_{2_i})(k_{3_j}+k_{4_j})} \langle k_1, k_2 | \, k_3, k_4 \rangle}$$

$$\boxed{\left\langle k_1, k_2 \,\middle|^{\star \mathrm{V}}\, k_3, k_4 \right\rangle = \mathrm{e}^{\theta(k_{1-}+k_{2-})(k_{3+}+k_{4-})} \langle k_1, k_2 | \, k_3, k_4 \rangle}$$

The expression for the inner product can be concisely written as:

$$\boxed{{}_{\star}\left\langle k_1, k_2 \,\middle|^{\star}\, k_3, k_4 \right\rangle_{\star} = \langle 0|\, a_{k_1} \star a_{k_2} \star a^{\dagger}_{k_3} \star a^{\dagger}_{k_4}\, |0\rangle}$$

$$\boxed{= \mathrm{e}^{-\sum_{a<b} k_a \bullet k_b} \langle k_1, k_2 |\, k_3, k_4 \rangle}$$

Which is a rather “natural” expression

We are now ready to calculate the S-matrix two particles elastic scattering (to one loop)

$$\boxed{S_{fi} = {}_{in\star}\left\langle f \,\overset{\star}{\Big|}\, i \right\rangle_{\star out} = {}_{in\star}\left\langle f \,\overset{\star}{\Big|}\, S \,\overset{\star}{\Big|}\, i \right\rangle_{\star in}}$$

The one-particle asymptotic state is

$$\boxed{|k\rangle_{in} = N_\star(k) a_k^\dagger |0\rangle_{in} = -N_\star(k) \frac{i}{\sqrt{(2\pi)^2 2\omega_k}} \int d^2x \mathrm{e}^{-ik\cdot x} \overset{\leftrightarrow}{\partial_0} \varphi_{in}(x) |0\rangle_{in}}$$

with $\boxed{N_\star(k)}$ a normalization factor to be determined for the Moyal and Wick-Voros cases

We assume

$$\boxed{\lim_{x^0 \to \pm\infty} \langle f|\varphi(x)|i\rangle = Z^{1/2} \langle f|\varphi_{\substack{out \\ in}}(x)|i\rangle}$$

$\boxed{Z}$ a renormalization factor

We obtain:

$$\boxed{S_{fi} = {}_{in\star}\left\langle k_1 k_2 \overset{\star}{\Big|} k_3 k_4 \right\rangle_{out\star} = \text{disconnected}}$$

$$\boxed{+\bar{N}_\star(k_1)\bar{N}_\star(k_2)N_\star(k_3)N_\star(k_4)(iZ^{-1/2})^2 \mathrm{e}^{-\sum_{a<b} k_a \bullet k_b}}$$

$$\boxed{\times \int \frac{\prod_a d^2x^a}{\sqrt{(2\pi)^2 2\omega_{k_a}}} \mathrm{e}^{-ik_a x^a} (\partial_\mu^2 + m^2)_a G(x^1, x^2, x^3, x^4)}$$

where $\boxed{G(x^1, x^2, x^3, x^4)}$ is the four-point Green's function.

to fix the normalization of the asymptotic states compute the scattering amplitude for one particle going into one particle, at zeroth order. Up to the undeformed normalization factors $\boxed{N(p_a)}$, this has to give a delta function:

$$\boxed{N^*(k)N(p)\delta^2(k-p) = N_\star^*(k)N_\star(p) \quad {}_{in\star}\left\langle k \overset{\star}{\Big|} p \right\rangle_{out\star}}$$

$$\boxed{= N_\star^*(k)N_\star(p)\mathrm{e}^{-k\bullet p} \; {}_{in}\langle k|\, p\rangle_{out} = N_\star^*(k)N_\star(p)\mathrm{e}^{-k\bullet p}\delta^2(k-p)}$$

from which we obtain

$$\boxed{N_{\star_\mathrm{M}}(p) = N(p)}$$

$$\boxed{N_{\star_\mathrm{V}}(p) = \mathrm{e}^{-\frac{\theta}{4}|\vec{p}|^2} N(p)}$$

For the planar case we have

$$\boxed{S_{fi_{\star P}}(k_1,..,k_4) = \bar{N}(k_1)\bar{N}(k_2)N(k_3)N(k_4)\Pi_a \mathrm{e}^{\frac{\theta}{4}|\vec{k}_a|^2}}$$

$$\boxed{\mathrm{e}^{-\sum_{a<b} k_a \bullet k_b} \int \Pi_a \frac{d^2x^a}{\sqrt{(2\pi)^2 2\omega_{k_a}}} \mathrm{e}^{-ik_a x^a} \int \Pi_a \frac{d^2 p_a}{\sqrt{(2\pi)^2 2\omega_{p_a}}} \mathrm{e}^{ip_a x^a}(-p_a^2 + m^2)}$$

$$\boxed{\int \mathrm{d}q \frac{\mathrm{e}^{\sum_{a\leq b} p_a \bullet p_b}}{(q^2 - m^2)((p_1 + k_2 - q)^2 - m^2)\prod_{a=1}^4 (p_a^2 - m^2)} \delta\left(\sum_{a=1}^4 p_a\right)}$$

The propagators of the external legs and the exponent cancel in the $\boxed{x}$ integration so that we are left with the usual undeformed expression

$$\boxed{S_{fi_{\star P}}(k_1,..,k_4) = S_{fi}(k_1,..,k_4)}$$

The nonplanar case is different form the undeformed case:

$$\boxed{S_{fi_{\star NP}}(k_1,..,k_4) = \bar{N}(k_1)\bar{N}(k_2)N(k_3)N(k_4)\Pi_a e^{\frac{\theta}{4}|\vec{k}_a|^2}}$$

$$\boxed{e^{-\sum_{a<b} k_a \bullet k_b} \int \Pi_a \frac{d^2x^a}{\sqrt{(2\pi)^2 2\omega_{k_a}}} e^{-ik_a x^a} \int \Pi_a \frac{d^2p_a}{\sqrt{(2\pi)^2 2\omega_{p_a}}} e^{ip_a x^a}(-p_a^2 + m^2)}$$

$$\boxed{\int dq \frac{e^{\sum_{a\leq b} p_a \bullet p_b + E_a}}{(q^2 - m^2)((p_1 + k_2 - q)^2 - m^2)\prod_{a=1}^4 (p_a^2 - m^2)} \delta\left(\sum_{a=1}^4 p_a\right)}$$

After integrating the previous cancellations still hold, but but we are left with the exponential of $\boxed{E_a}$ which doesn't simplify.

It is an imaginary phase, and it has the same expression in the Moyal and Wick-Voros case. It depends on the $\boxed{q}$ variable, therefore it gets integrated and modifies the IR and UV behaviour of the loop: this is the correction responsible for the UV/IR mixing

Conclusions

We have investigated two different but equivalent noncommutative $\boxed{\star}$ product. With the prejudice that they should give the same "physical predictions"

We have found that they give different Green's functions and free propagators

We have also found that if the calculations of the S-matrix are made in a controlled way, deforming all products, then the result is indeed the same for the two noncommutative theories, but it is different from the undeformed case. To achieve this we had to twist the asymptotic states, the way we put together representations, the product among creation and annihilation operators, the inner product,and even the normalization, all not in a random way, but following a precise procedure.

We can quote the Duca di Salina in Tomasi di Lampedusa's novel *Il Gattopardo*:

Or Luchino Visconti's movie *The Leopard* with Burt Lancaster

Change everything so that nothing changes