Noncommutative Geometry

a general (and generic) review

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Let me start with a definition of Geometry

For which I can think of no higher authority than **Wikipedia**

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Geometry (Greek $\gamma\varepsilon\omega\mu\varepsilon\tau\rho\iota\alpha$; geo = earth, metria = measure) is a part of mathematics concerned with questions of size, shape, and relative position of figures and with properties of space.

And this geometry has served us quite well for a few millennia

In this talk I will give a non-technical, non-rigorous, review of the efforts that physicist have doing to use a generalization of the usual geometrical objects. I will not go into any detail and will not go into any depth.

The need for a noncommutative geometry

Classical mechanics can be seen as the study of the geometry of phase space (or position momentum space). Given an initial position of a system, the classical dynamics describes its evolution.

We can have the case of constrained mechanics, the infinite dimensional case, and also the relativistic case

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All this changes dramatically with quantum mechanics

A quick way to see that in quantum mechanics the concept of point (of phase space) is not valid is given by the Heisenberg Microscope

The idea is that to "see" something small, of size of the order of Δx, we have to send a "small" photon, that is a photon with a small wavelength λ, but a small wavelength means a large momentum $p = h/\lambda$. In the collision there will a transfer of momentum, so that we can capture the photon. The amount of momentum transferred is uncertain.

In quantum mechanics a point in phase space is an untenable concept because of the Heisenberg uncertainty principle:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

We know what has happened. The observables, which in classical mechanics are commutative functions on the phase space, have become noncommutative operators on the Hilbert space of wave functions

A (pure) state in classical mechanics is a point of phase space, an observable is something which gives a real number for each state (the value of the function on the point)

A (pure) state in quantum mechanics is a vector on the Hilbert space, an observable is something which gives a real number for each state (the expectation value of the operator)

The difference is noncommutativity

The information on the (classical) phase space is encoded in the (commutative) algebra of observables, i.e. in the functions on the space

There is a theorem (Gelfand-Naimark) which demonstrates a complete equivalence between commutative C^*-algebras and Hausdorff topological spaces

A Hausdorff space is one for which points are separable. A C^*-algebra is an associative algebra with a norm and a complex conjugation

Given an Hausdorff space it is always possible to construct a commutative C^*-algebra: continuous complex valued functions.

Remarkably the converse is also true, an arbitrary C^*-algebra is always the algebra of continuous complex valued functions on some Hausdorff space. The points of the space are the pure states of the algebra, the topology is given by convergence.

For a commutative algebra pure states, irreducible one-dimensional representations and maximal ideals all coincide, and the same topology can be constructed from either of these sets.

This duality has led some people, starting with Von Neumann, but principally Alain Connes, to an attempt to transcribe all properties of ordinary spaces in algebraic terms

Thus the emphasis in the description of geometry switches from points to fields

The topology is encoded by the algebra, which can always be represented as operators on some Hilbert space (loosely speaking, every algebra is a matrix algebra, possibly infinite dimensional)

The metric structure is encoded in a (generalized) Dirac operator D, which "knows" about the metric, and is used to build the differential calculus (forms). Integrals become traces of operators with the inverse of D playing the role of the measure

What if the algebra is noncommutative?

Noncommutative Spaces

If the algebra is noncommutative the identification of points with pure states (or irreducible representations) fail. Often the Hausdorff topology gives a single points

Nevertheless the geometry information about the space is encoded in the noncommutative algebra, and possibly in some further objects like the D operator above

If we succeed in transcribing objects of ordinary geometry in algebraic terms, then the generalization is "simply" done just assuming that the algebra is noncommutative

Quantum phase space is a noncommutative space, but what are its relations with the classical space?, With its structures?

$\boxed{\text{Deformation of spaces}}$

Take the algebra of classical observables, functions multiplied with the commutative product, and introduce a deformed (Gronewöld-Moyal) product:

$$(f \star g)(x,p) = f e^{\frac{i\hbar}{2}\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x}} g = fg + \frac{i\hbar}{2}(\partial_x f \partial_p g - \partial_p f \partial_x g) + O(\hbar^2)$$

So that to first order in $\hbar$

$$f \star g - g \star f = i\hbar\{f, g\}$$

The commutator is a deformation of the Poisson bracket, in the limit $\hbar \to 0$ one finds again the classical structure

The noncommutative structure of spacetime

So far we have been discussing the noncommutativity of phase space. In quantum mechanics however configuration space is still an ordinary space

Is it legitimate to expect the usual geometry to hold to all scales?

There are several arguments which indicate physical reasons for which it should not be so

Just to mention one (Doplicher-Fredenhagen-Roberts) which is a variation of the Heisenberg microscope, at the same caricature level I used before:

In order to “measure” the position of an object, and hence the “point” in space, one has use a very small probe, and quantum mechanics forces us to have it very energetic, but on the other side general relativity tells us that if too much energy is concentrated in a region a black hole is formed.

The scale at which this happens is of the order of Planck’s length

$$\ell_P = \sqrt{\frac{G\hbar}{c^3}} = 1.6\ 10^{-33}\ cm.$$

This is the region in which the theory to use is Quantum Gravity. Unfortunately a theory we do not yet have

In fact the two problems are related. A quantum gravity theory needs spacetime to be a different object from the one used in classical geometry

For example in loop quantum gravity 3-space is directly quantized and the geometry used there (spin networks) is certainly different from the classical one

Also in string theory spacetime undergoes changes

It is not anymore a given starting point but for example its dimensions emerge from the quantization of a conformal two-dimensional field theory

Interacting strings are described by the insertion of vertex operators on the worldsheet

At ultra high energy the structure of spacetime is again a (still somewhat mysterious) object in which ordinary spacetime has undergone strong transformations (M theory)

These considerations have led several people to consider space-time as a noncommutative space, and to use the tool of non-commutative geometry

Historically the first appearance of Noncommutative Geometry in a physics paper was Witten's Open String Field Theory (1986)

String fields are seen as maps from a string configuration in space into complex numbers, with an enormous gauge symmetry (reparametrisation). After gauge fixing the role of differential is played by the BRS operator

Then in the 90's there was Connes' approach to the Standard Model

Connes' approach to the standard model

The project is to transcribe electrodynamics on an ordinary manifold using algebraic concepts: The algebra of functions, the Dirac operator, the Hilbert space and some added operators (Chirality and charge conjugation). One can then write the action in purely algebraic terms.

Then the machinery can be applied to noncommutative space, or in general to other algebras.

Remarkably, if one applies this to the algebra of functions valued in diagonal 2×2 matrices one finds the Lagrangian of the Higgs breaking of a $U(1) \times U(1) \to U(1)$ theory, in which the Higgs is the "vector" boson corresponding to the internal degree of freedom.

In this case the space is only “almost” noncommutative, in the sense that there still is an underlying spacetime, the noncommutative algebra describing space is said to be Morita equivalent to a commutative algebra

For the full standard the algebra is a tensor product $\mathcal{A} = C(\mathbf{R}^4) \otimes \mathcal{A}_F$, with $\mathcal{A}_F$ a finite matrix algebra of 3×3 matrices, quaternions (which are matrices of the kind $a^\mu \sigma_\mu$) and complex numbers corresponding to $SU(3), SU(2)$ and $U(1)$ respectively.

The information about mass and Cabibbo mixing are encoded in the D operator

The aim is not to predict the Lagrangian of standard model (taken as input) but to find a noncommutative geometry which describes the standard model

The model, especially in its last version (Chamseddine-Connes-Marcolli) has some predictive power (mass of the Higgs), but it is inherently classical, and once a Lagrangian is written, renormalization is performed in the usual way

Impulse to study noncommutative spaces in physics came again from Strings Frohlich-Gawedski, Landi-FL-Szabo, Seiberg-Witten when it turned out that, in some limit, the vertex operators of a string theory show the behaviour given by noncommutative coordinates

In the spirit of what I said before one can threat a noncommuting space deforming the algebra of functions with a $\star$ product similar to the one introduced in quantum mechanics, with $\hbar$ replaced by an antisymmetric matrix θ:

$$f \star g = f e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}} g$$

In this way we encode the noncommutativity of spacetime in the deformation of the algebra

Noncommutative Field Theory

Deform of a commutative theory with the presence of a star product among the fields. For example

$$S = \int d^d x \partial_\mu \varphi \star \partial^\mu \varphi + m^2 \varphi \star \varphi + \frac{g^2}{4!} \varphi \star \varphi \star \varphi \star \varphi$$

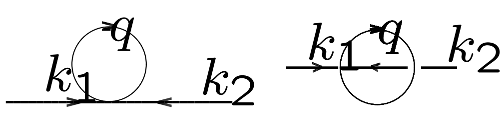
For the Grönewold-Moyal product the $\star$ on the first two terms is redundant because $\int d^d x f \star g = \int d^d x f g$

What physics comes out of these theories?

The free theory is unchanged because of the integral property. But the vertex gets a phase. For the example $\varphi^{\star 4}$:

$$V = (2\pi)^4 g\delta^4\left(\sum_{a=1}^{4} k_a\right)\prod_{a<b}{}^{-\frac{i}{2}\theta^{\mu\nu}k_{a_\mu}k_{b_\nu}}$$

The vertex is not anymore invariant for exchange of the momenta (only for cyclic permutations), and causes a difference between planar and nonplanar diagrams



A consequence of this is Ultraviolet/Infrared Mixing Minwalla-Seiberg-Van Raamnsdong. The phenomenon for which some ultraviolet divergences disappear, just to reappear as infrared divergences

If we take seriously the fact that the world is described by this kind of noncommutative field theory which are the consequences? How do we measure $\theta^{\mu\nu}$, a quantity of the order of ℓ_P^2?

At this level, and as suggested by string theory, $\theta^{\mu\nu}$ is a background quantity, which selects two directions in space (analog of electric and magnetic fields). Their presence breaks Lorentz invariance and the noncommutativity will have left its imprinting in the early universe, and its consequences are thereafter frozen by inflation

Direct accelerator measurements are more difficult because the earth rotation washes up the effect. But one can look for otherwise forbidden processes

The problem is that the Moyal product is made for flat coordinates. The construction of associative deformed products is not simple (Kontsevich has won a field medal building them). One cannot simply substitute, say, the partial derivatives in the definition with covariant derivatives.

Nevertheless something has been done (Chu-Greene-Shiu, Brandemberger, FL-Mangano-Miele-Peloso). In our work we considered the field theory of a field which causes inflation to be deformed by a star product

It is not easy however to distinguish predictions coming from these kind of theories from other breakings of Lorentz invariance

But, given that we want to use noncommutativity of spacetime, are we sure we are using the right one? And what about the breaking of Lorentz symmetry in a fundamental theory?

A deformation of spacetime may require a deformation of symmetries. **Quantum Groups and Hopf Algebras**

A Lie group is a manifold, and therefore it is a topological space, described by its commutative algebra of functions. It has however added structure: it makes sense to multiply "points", there is an identity, an inverse of every point.

This structure is encoded in the algebra of functions as a coproduct, which from a function of one variable gives a function of two variables:

$$\Delta(f)(g_1, g_2) = f(g_1 g_2)$$

The Lie algebra level (infinitesimal transformations, or differential operators), the coproduct in the group induces a coproduct in the algebra

$$\Delta(L) = L \otimes I + I \otimes L$$

Which is the Leibnitz rule when L, element of the Lie algebra is seen as a differential operator. This (and other structures) gives the structure of a Hopf algebra

A quantum group is what we obtain when the algebra of functions on the group becomes noncommutative. It is then necessary to deform commutation relations and/or coproducts.

There is a Hopf algebra which is causing great interest:

θ-Poincaré

Consider the symmetry to be a twisted quantum symmetry (Wess and the Münich group: Aschieri, Blohmann, Dimitriević, Meyer, Schupp, Chaichian-Kulish-Nishijima-Tureanu, Oeckl, Majid, Drinfeld . . .)

Consider the usual action of the Lie algebra L of differential operators on the algebra $\mathcal{A}$ of functions with the usual commutative product

The usual product can be seen as a map from $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$

$$m_0(f \otimes g) = fg$$

with pointwise multiplication

The Leibnitz rule imposes a coalgebra structure of the Lie algebra:

$$\ell(fg) = \ell(f)g + f\ell(g) = m_0(\Delta(L)(f \otimes g))$$

where ℓ is a generic first order differential operator

$$\Delta : L \to L \otimes L$$

$$\Delta(\ell) = \ell \otimes 1 + 1 \otimes \ell$$

The coproduct tells how to put together representations, and how an operator acts on two copies of the module.

Consider the Moyal product as follows

$$(f \star g)(x) = m_0[\mathcal{F}^{-1} f \otimes g] \equiv m_\theta[f \otimes g]$$

where $$m_0(f \otimes g) = fg$$

is the ordinary product and

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_{x\mu}\otimes\partial_{y\nu}} = e^{-\frac{i}{2}\theta(\partial_{x_0}\otimes\partial_{y_1}-\partial_{x_1}\otimes\partial_{y_0})}$$

is called the twist.

The noncommutative product is obtained first twisting the tensor product, and then using the ordinary product.

With the twist we have to revise the Leibnitz rule:

$$\partial_\mu(f \star g) = m_\theta \Delta_\theta(f \otimes g) = m_0 \Delta(\partial_m u)(\mathcal{F}^{-1}(f \otimes g))$$

where

$$\Delta_\theta = \mathcal{F} \Delta \mathcal{F}^{-1}$$

The algebra structure remains unchanged, what changes is the coalgebra structure, that is the way to “put together representations”.

counit and antipode remain unchanged.

We have this deformed the coalgebra structure of the Poincaré Lie algebra. In particular:

The Lie algebra structure (commutators) is not changed. What changes is the coalgebra, at the level of the Lorentz group

$$\Delta_{\mathcal{F}}(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu$$

$$\Delta_{\mathcal{F}}(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta}\left((\eta_{\alpha\mu}P_\nu - \eta\alpha\nu P_\mu) \otimes P_\beta + P_\beta\left(\eta_{\beta\mu}P_\nu - \eta\beta\nu P_\mu\right)\right)$$

The fact that the algebra is the same means that we can still use the casimirs and the representations of the usual algebra, with thus concepts of mass, spin etc.

The twisted framework for noncommutative field theory is still under investigation, and is not free from controversies

We have changed the tensor product, and therefore one should twist all products in an appropriate way

Nevertheless there are already attempts at prediction, both in the gravitational framework, in the form of a deformed Einstein-Hilbert action (Wess et al.), or in the changes of statistics due to the twist.

But we are probably still lacking a “canonical” procedure to understand the twist

Another possibility could be κ-Minkowki. This is the homogenous space of the κ-Poincaré quantum group, and it is characterized by the commutation relations

$$[x_i, x_0] = i\lambda x_i, \quad [x_i, x_j] = 0$$

The commutation relations for κ-Poincaré are:

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[M_i, P_j] &= i\epsilon_{ijk} P_k \\
[M_i, P_0] &= 0 \\
[N_i, P_j] &= -i\delta_{ij}\left(\frac{1}{2\lambda}(1 - e^{2\lambda P_0}) + \frac{\lambda}{2}P^2\right) + i\lambda P_i P_j \\
[N_i, P_0] &= iP_i \\
[M_i, M_j] &= i\epsilon_{ijk} M_k \\
[M_i, N_j] &= i\epsilon_{ijk} N_k \\
[N_i, N_j] &= -i\epsilon_{ijk} M_k
\end{aligned}
$$

All these commutation relations become the standard ones for $\lambda \to 0$. The bicrossproduct basis is peculiar as κ-Poincaré acts *covariantly* on a space that is necessarily deformed and noncommutative. This is a consequence of the non cocommutativity of the coproduct which, always in the bicrossproduct basis, reads:

$$\begin{aligned}
\Delta P_0 &= P_0 \otimes 1 + 1 \otimes P_0 \\
\Delta M_i &= M_i \otimes 1 + 1 \otimes M_i \\
\Delta P_i &= P_i \otimes 1 + e^{\lambda P_0} \otimes P_i \\
\Delta N_i &= N_i \otimes 1 + e^{+\lambda P_0} \otimes N_i + \lambda \varepsilon_{ijk} P_j \otimes M_k
\end{aligned}$$

The Casimir of this quantum group provide a deformation of the Energy-Momentum dispersion relation and this could be used to explain γ-ray bursts (Amelino-Camelia). The problem is that, being the commutation relations nonlinear, nonlinear changes of coordinates are allowed, and therefore these dispersion relations become basis-dependent.

Is it possible to draw conclusions?

I knowingly avoided to give details of the physical predictions of the various kinds of noncommutative geometry. Not because there are none, but because I preferred to give an overview of the tool, not of the artifacts. Each of those would deserve a seminar.

My personal conclusion is that at the Planck scale there should be a noncommutative structure, and that we are developing a set of tools which are likely to be the right ones to describe physics at the Planck length.

Noncommutative geometry should probably complement a more general theory, strings, loop quantum gravity, . . .

Fortunately we can expect some input from experiments and observations: LHC, Planck, cosmic rays

Things I did not even mention. . .

- Noncommutative gauge theories, Seiber Witten map
- The noncommutative geometry of string theories (Vertex operators, Hilbert space of string states, duality etc.
- The noncommutative geometry of loop quantum gravity (spin networks, projective limits etc.)
- Fuzzy spheres and discs, field theories on fuzzy spaces
- Noncommutative (non Hausdorff) lattices
- Noncommutative spheres, tori and other noncommutative spaces
- Matrix models