Twisting all the way

from Classical Mechanics to Quantum Fields

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I need not motivate to this audience the need for a Noncommutative Geometry of spacetime

The implementation of noncommutativity via the twist gives the possibility to maintain symmetries of spacetime also in the presence of the Grönewold-Moyal product, albeit in a deformed guise

Still there is discussion as to the correct way to implement this product at the quantum level

We take a step backward and start discussing the twist in classical mechanics

Our guiding guiding is the following:

Whenever you see a product: Twist it!

Given an associative algebra, which we may think as representing spacetime, and a Lie algebra Ξ, acting on it, we introduce the twist, an invertible element of $U\Xi \otimes U\Xi$ (Sweedler notation)

$$\mathcal{F}^{-1} = \bar{\mathrm{f}}^{\alpha} \otimes \bar{\mathrm{f}}_{\alpha}$$

$$\mathcal{F} = \mathrm{f}^{\alpha} \otimes \mathrm{f}_{\alpha}$$

For a bilinear map (in particular a product if $X = Y = Z$)

$$\mu : X \otimes Y \to Z$$

where we assumed that X and Y are $U\Xi$ modules.

Now define the twisted map

$$\mu_\star : \bar{\mathsf{f}}^\alpha(X) \times \bar{\mathsf{f}}_\alpha(Y) \to Z$$

where by $\bar{\mathsf{f}}^\alpha(X)$ we indicate the appropriate representation of the twist

We now consider the “Moyal” twist, whose representation on the space of functions on $\mathbf{R}^n$ is:

$$\mathcal{F} = \mathrm{e}^{-\frac{i\theta^{ij}}{2}\partial_i \otimes \partial_j}$$

So that the Lie algebra Ξ is that of translations

If all X, Y and Z are $C(\mathbf{R}^n)$ the pointwise product is deformed into the Moyal product: $f \star g = \bar{\mathrm{f}}^{\alpha}(f) \cdot \bar{\mathrm{f}}_{\alpha}(g)$

Define also the universal $\boxed{R}$ matrix:

$$\mathcal{R} := \mathcal{F}_{21}\mathcal{F}^{-1} := \mathsf{R}^{\alpha} \otimes \mathsf{R}_{\alpha} = \mathcal{F}^{-2}$$

It measures the noncommutativity of the product:

$$f \star g = \bar{\mathsf{R}}^{\alpha}(g) \star \bar{\mathsf{R}}_{\alpha}(f)$$

with $\mathcal{R}^{-1} := \ \bar{\mathsf{R}}^{\alpha} \otimes \bar{\mathsf{R}}_{\alpha}$

Classical Mechanics

Phase space $\mathbf{R}^{2n}$ with coordinates x^m, p_m and another product: **Poisson bracket**

$$\{f, g\} := \partial_m f \ \bar{\partial}^m g - \bar{\partial}^m f \ \partial_m g$$

$$\partial_m := \frac{\partial}{\partial x^m} \quad ; \quad \bar{\partial}^m := \frac{\partial}{\partial p_m}$$

The bracket is antisymmetric, satisfies Leibniz and Jacobi. To it we associate a derivation $X_f \equiv \{f, \cdot\}$ and hence a vector field. So the bracket is really a Lie derivative

$$\{f, g\} = \mathcal{L}_{X_f} g$$

The vector fields form a Lie algebra

$$[\mathcal{L}_{X_f}, \mathcal{L}_{X_g}] = \mathcal{L}_{X_{\{f,g\}}}$$

Time evolution is given by Hamiltonian $\dot{f} = -\{H, f\} = -\mathcal{L}_{X_H} f$

And we know how to implement all other symmetries, translations, rotations etc.

A classical system is invariant under a Lie algebra $\mathcal{G}$ with generators X_i if

$$\mathcal{L}_{X_i} H = 0$$

Twist!

$$\{f, g\}_\star = \{\bar{\mathsf{f}}^\alpha(f), \bar{\mathsf{f}}_\alpha(g)\}$$

A simple calculation shows

$$\{f, g\}_\star = \partial_m f \star \bar{\partial}^m g - \bar{\partial}^m f \star \partial_m g$$

Antisymmetry, Leibniz and Jacobi now hold in twisted form

$$\{f, g\}_\star = -\{\bar{\mathsf{R}}^\alpha(g), \bar{\mathsf{R}}_\alpha(f)\}_\star \qquad \{f, g \star h\}_\star = \{f, g\}_\star \star h + \bar{\mathsf{R}}^\alpha(g) \star \{\bar{\mathsf{R}}_\alpha(f), h\}_\star$$

$$\{f, \{g, h\}_\star\}_\star = \{\{f, g\}_\star, h\}_\star + \{\{f, \bar{\mathsf{R}}^\alpha(h\}_\star\}, \bar{\mathsf{R}}^\alpha(g)\}_\star$$

The Lie algebra of vector fields becomes a non cocommutative Hopf algebra

$$\Delta_\star(X) = X \otimes 1 + \bar{\mathsf{R}}^\alpha \otimes \bar{\mathsf{R}}_\alpha(X)$$

The twisted Leibniz rule can be read as

$$X^\star(g \star h) = \mu_\star \circ \Delta_\star(X)(g \otimes h)$$

You may be used to a different Hopf algebra structure with $\Delta_{\mathcal{F}} = \mathcal{F}\Delta_0\mathcal{F}^{-1}$.
The two Hopf algebras are isomorphic

This twisted classical mechanics has interesting features. Defining naturally

$$\dot{f} = -\{H, f\}_\star = -\mathcal{L}^\star_{X_H} f$$

we see that, for an Hamiltonian of the kind $H = p^2 + V(x)$ the Poisson brackets of x and p do not change

$$\dot{x}^m = -\{H, x^m\}_\star = -\{H, x^m\} \qquad \dot{p}_m = -\{H, p_m\}_\star = -\{H, p_m\}$$

This is not true for other functions. Take two-dimensional angular momentum $L = \epsilon_{ij} x^i p^j$ and the harmonic oscillator

$$H = \frac{1}{2}(x^i \star x^j \delta_{ij} + p_i \star p_j \delta^{ij}) = \frac{1}{2}(x^i x^j \delta_{ij} + p_i p_j \delta^{ij})$$

then

$$\dot{L} = \frac{i}{2}\epsilon_{ij}\theta^{ij} = i\theta$$

angular momentum in not a constant of the motion, and the Hamiltonian is not rotationally invariant!

This should come as no surprise, we are to implement a symmetry for rotation. They do not form a Hopf subalgebra of the Hopf algebra of vector fields

Fields

The degrees of freedom are $\Phi(x), \Pi(x)$ with (equal time) Poisson brackets among functionals:

$$\{F, G\} = \int_{\mathbb{R}^{n-1}} \mathrm{d}^{n-1}x \left(\frac{\partial F}{\partial \Phi}\frac{\partial G}{\partial \Pi} - \frac{\partial F}{\partial \Pi}\frac{\partial G}{\partial \Phi}\right)$$

coming from the equal time

$$\{\Phi(x), \Phi(y)\} = \{\Pi(x), \Pi(y)\} = 0 \qquad \{\Phi(x), \Pi(y)\} = \delta(x - y)$$

Twist!

$$\{F, G\}_\star = \{\bar{\mathsf{f}}^\alpha(F), \bar{\mathsf{f}}_\alpha(G)\}$$

We have to qualify the action of the twist on the fields: Lift of the action of ∂_μ on $\mathbf{R}^n$ to ∂^*_μ acting on functionals of the fields:

$$\partial^*_\mu G = \int \mathrm{d}^{n-1}y \frac{\partial G}{\partial \Phi(y)} \partial_\mu \Phi(y)$$

we hence define the lift of the twist

$$\mathcal{F}^* = \mathrm{e}^{\frac{i}{2}\theta^{ij} \int \mathrm{d}^{n-1}x \left(\partial_i \Phi \frac{\partial}{\partial \Phi(x)} + \partial_i \Pi \frac{\partial}{\partial \Pi(x)}\right) \otimes \int \mathrm{d}^{n-1}y \left(\partial_j \Phi \frac{\partial}{\partial \Phi(y)} + \partial_j \Pi \frac{\partial}{\partial \Pi(y)}\right)}$$

Calculate the deformed bracket among the fields:

$$\{\Phi(x), \Pi(y)\}_\star = \{\bar{\mathsf{f}}^\alpha(\Phi(x)), \bar{\mathsf{f}}_\alpha(\Pi(y))\} =$$

$$\delta(x-y) - \frac{i}{2}\theta^{ij}\left\{\int \mathrm{d}z \partial_i^* \Phi(z)\delta(x-z), \int \mathrm{d}w \partial_j^* \Pi(w)\delta(y-w)\right\} + O(\theta^2) =$$

$$\delta(x-y) - \frac{i}{2}\theta^{ij}\partial^*_{y^j}\partial^*_{x^i}\delta(x-y) + O(\theta^2)$$

The second term in the third line of the above equation vanishes because of symmetry, and for similar reasons the others vanish as well. Therefore also in the field theoretical case the Poisson bracket among coordinates is unchanged.

Functionals will however have a deformed dynamics, and this may have consequences for theories with solitons etc. Leave it for another day and proceed to:

Quantization

We now proceed to the canonical quantization of the fields

expansion of the classical fields in Fourier modes using the Lorentz invariant measure $E_p = \sqrt{m^2 - \vec{p}^2}$

$$\Phi(x) = \int \frac{\mathrm{d}^2 p}{(2\pi) 2E_p} \left(a(\vec{p}) \mathrm{e}^{-ipx} + a^*(\vec{p}) \mathrm{e}^{ipx} \right)$$

$$\Pi(x) = \int \frac{\mathrm{d}^2 p}{(2\pi)} \left(-iE_p a(\vec{p}) \mathrm{e}^{-ipx} + iE_p a^*(\vec{p}) \mathrm{e}^{ipx} \right)$$

the twisted relations can be inverted to give

$$a(\vec{p}) = \frac{1}{2\pi} \int \mathrm{d}^2 x \left(E_p \Phi(x) \mathrm{e}^{ipx} + i \Pi(x) \mathrm{e}^{ipx} \right)$$

$$a^*(\vec{p}) = \frac{1}{2\pi} \int \mathrm{d}^2 x \left(E_p \Phi(x) \mathrm{e}^{-ipx} - i \Pi(x) \mathrm{e}^{-ipx} \right)$$

and this gives the Poisson bracket

$$\{a(\vec{p}), a^*(\vec{q})\} = -2iE_p \delta(\vec{p} - \vec{q})$$

As in the case of the fields, with a similar calculation, it is possible to see that the twisted Poisson bracket is equal to the untwisted one

$$\{a(\vec{p}), a^*(\vec{q})\}_\star = -2iE_p \delta(\vec{p} - \vec{q})$$

To quantize we equate the twisted commutator to $i\hbar$ times the twisted Poisson bracket.

To define the twisted commutator we use the rule which has served us so far:

$$[\widehat{F}, \widehat{G}]_\star = [\bar{\mathsf{f}}^\alpha(\widehat{F}), \bar{\mathsf{f}}_\alpha(\widehat{G})]$$

Using the fact that $\mathcal{R} = \mathcal{F}^{-2}$ it is possible to see that

$$[\widehat{F}, \widehat{G}]_\star = \widehat{F} \star \widehat{G} - \bar{\mathsf{R}}^\alpha(\widehat{G}) \star \bar{\mathsf{R}}_\alpha(\widehat{F})$$

The commutator is the difference between the product of two elements of an algebra minus the product of the two elements taken in the reverse order

In the twisted algebra however the inversion is made via the inverse $\mathcal{R}$ matrix

Applying this to the fundamental fields:

$$[\widehat{\Phi}(x), \widehat{\Pi}(y)]_\star = \widehat{\Phi}(x) \star \widehat{\Pi}(y) - \bar{\mathsf{R}}^\alpha(\widehat{\Pi}(y)) \star \bar{\mathsf{R}}_\alpha(\widehat{\Phi}(x))$$

where the product of two fields at different points is defined as always by the twist:

$$\widehat{\Phi}(x) \star \widehat{\Pi}(y) = \bar{\mathsf{f}}^\alpha(\widehat{\Phi}(x))\bar{\mathsf{f}}_\alpha(\widehat{\Pi}(y))$$

Canonical quantization consists in setting the (star) commutator equal to the (star) Poisson bracket:

$$[\hat{a}(\vec{p}), \hat{a}^*(\vec{q})]_\star = i\hbar\{a(\vec{p}), a^*(\vec{q})\} = 2i\hbar E_p \delta(p-q)$$

The $\star$ commutator among the Fourier modes does not involve the product among the coordinates, but it involves the $\mathcal{R}$ matrix.

To see its action we have to represent the action of ∂_i on $\hat{a}(\vec{p})$.

Since these are Fourier modes, from the action of ∂_i^* on Φ, Π it results

$\partial_i \triangleright \hat{a}(\vec{p}) = i p_i a(\vec{p})$

Using again the fact that $\mathcal{R} = \mathcal{F}^{-2}$ it then results:

$$[\widehat{a}(\vec{p}), \widehat{a}^*(\vec{q})]_\star = \widehat{a}(\vec{p})\widehat{a}^*(\vec{q}) - \mathrm{e}^{i\theta^{ij}p_i q_j}\widehat{a}^*(\vec{q})\widehat{a}(\vec{p}) = 2\hbar E_p \delta(\vec{p} - \vec{q})$$

The above relation has been introduced already in the literature (Bal et al. Fiore-Wess...), with phenomelogical consequences, and causing also some controversies

We find it coming from a well defined coherent procedure defined from first principles, Basically it is a calculation, and this lends support to this ealrlier work

Conclusions

It is probably time Noncommutative Geometry starts confronting itself with experiments

To do this we have to build a field theory, and among the possible choices, only some sort of canonical procedure, starting from first principle, and respecting symmetries (also in a deformed form) can ensure coherence of the theory

We hope to have given a contribution in this direction