Twisted Conformal Symmetry in Noncommutative Two-Dimensional Quantum Field Theory

Fedele Lizzi

With S. Vaidya & P. Vitale

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Belavin, Polyakov Zamolodchikov in Moyal sauce

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If we were to make a “hit parade” of the most famous symmetries, a list could probably go:

1. Rotational 518K

2. Conformal 143K

3. Lorentz/Poincaré 10K

4. SU(3) 3K

Source: google scholar

Conformal symmetry ranks second in this (totally unscientific) classification.

The reason is that in two dimensions conformal symmetry is a most interesting infinite dimensional symmetry

It is the symmetry of the world sheet of string theory, just to name an application.

Consider a two dimensional theory with Minkowski signature:

$$x^{\pm} = x^0 \pm x^1$$

A conformal transformation is just a purely "holomorphic" or "antiholomorphic" tranformation:

$$x^{\pm} \longrightarrow u^{\mp}(x^{\pm})$$

A treatment using $z = x^0 + iz^1, \bar{z} = x^0 - ix^1$ is also standard

The generators of this symmetry are:

$$\ell_n^{\pm} = -(x^{\pm})^n \partial_{\pm}$$

and generate the classical Virasoro-Witt algebra

$$[\ell_n^{\pm}, \ell_n^{\pm}] = (m-n)\ell_{n+m}^{\pm} \quad , \quad [\ell_n^{+}, \ell_m^{-}] = 0$$

One can write a conformal invariant action:

$$S = \int d^2x \; \partial_+ \varphi * \partial_- \varphi$$

The classical solutions are fields split in “left” and ‘right” movers

$$\varphi = \varphi_+(x^+) + \varphi_-(x^-)$$

Quantizing the theory we have an expansion:

$$\phi(x^0, x^1) = \int_{-\infty}^{\infty} \frac{dk^1}{4\pi k_0} \left(a(k) e^{-ik^\mu x_\mu} + a^\dagger(k) e^{ik^\mu x_\mu} \right)$$

Using $k^0 = k_0 = |k^1|$ we have:

$$\phi(x^+, x^-) = \int_{-\infty}^{0} \frac{dk^1}{4\pi|k^1|} \left(a(k) e^{-i|k^1|x^+} + a^\dagger(k) e^{i|k^1|x^+} \right)$$

$$+ \int_{0}^{\infty} \frac{dk^1}{4\pi|k^1|} \left(a(k) e^{-ik^1 x^-} + a^\dagger(k) e^{ik^1 x^-} \right)$$

where reality of the fields implies $a(-k) = a^\dagger(k)$.

The canonical commutation relations give:

$$[a(p), a(q)] = 2p\delta(p + q)$$

The quantum currents $J^{+}(x) = \partial^{+}\phi, \quad J^{-}(x) = \partial^{-}\phi$ generate two commuting U(1) Kaç-Moody algebras with opposite central charges.

$$\left[J^{\pm}(x), J^{\pm}(y)\right] = -\frac{i}{\pi}\partial_{\mp}\delta(x^{\mp} - y^{\mp})$$

This algebra gives the central extension of the Virasoro algebra

Quantum conformal invariance is proved showing that the components of the quantum stress-energy tensor generate the conformal algebra.

The quantum stress-energy tensor is the normal-ordered product

$$\Theta^{\pm\pm}(x) \propto : J^{\pm}(x) J^{\pm}(x) :$$

the other components vanish

normal ordering is defined:

$$: a(p)a(q) : = a(p)a(q) \text{ if } p < q$$

$$: a(p)a(q) : = a(q)a(p) \text{ if } p \geq q$$

The existence of Kaç-Moody quantum current algebras is a sufficient condition to ensure conformal invariance at the quantum level.

And also Noncommutative Geometry figures not so badly in the hit parade 9K (for both words)

So we can try to put together the two.

Unfortunately this is "impossible".

We will consider the noncommutative plane equipped with the usual Moyal product

$$(f * g)(x) = e^{\frac{i}{2}\theta(\partial_{x_0}\partial_{y_1} - \partial_{x_1}\partial_{y_0})} f(x) \cdot g(y)|_{x=y}$$

Where θ is a constant wit the dimensions of the square of a length

Conformal theories are scale invariant and a theory with a fundamental scale cannot be scale invariant.

The Virasoro generators have a different algebra:

$$\left[(x^{+})^{n+1}\partial_{+}, (x^{-})^{m+1}\partial_{-}\right]_{\star} \neq 0$$

The Virasoro generators do not respect Leibnitz rule

$$(x^{\pm})^{n+1}\partial_{\pm}(f \star g) \neq f \star (x^{\pm})^{n+1}\partial_{\pm} g + \left((x^{\pm})^{n+1}\partial_{\pm} f\right) \star g$$

or for that matter

$$(x^{\pm})^{n+1} \star \partial_{\pm}(f \star g) \neq f \star (x^{\pm})^{n+1} \star \partial_{\pm} g + \left((x^{\pm})^{n+1} \star \partial_{\pm} f\right) \star g$$

To say that a theory with a scale with the dimension of a length cannot be scale invariant is a little like saying that a theory with a constant of the dimensions of the angular momentum, $\hbar$, cannot be rotationally invariant...

As for the commutation, let us define the generators acting on functions as:

$$\ell_n^+(f) = (x^+)^{n+1} * \partial_+ f$$
$$\ell_n^-(f) = \partial_- f * (x^-)^{n+1}$$

Then the commutation relations are the usual ones.

We still have problems when we put functions together

The solution (Wess, Chaichian-Kulish-Nishijima-Tureanu), is to consider the symmetry to be a quantum symmetry

Consider the Moyal product as follows

$$(f * g)(x) = m_0[\mathcal{F} f \otimes g] \equiv m_\theta[f \otimes g]$$

where $m_0(f \otimes g) = fg$ is the ordinary product and

$$\mathcal{F} = e^{\frac{i}{2}\theta^{\mu\nu}\partial_{x_\mu}\otimes\partial_{y_\nu}} = e^{\frac{i}{2}\theta(\partial_{x_0}\otimes\partial_{y_1}-\partial_{x_1}\otimes\partial_{y_0})}$$

is called the twist.

We take the point of view that the Noncommutative plane is obtained twisting the tensor product

As a first consequence the coproduct, that is the way to put together the representations, is altered:

$$\Delta_\theta(\ell_n^+) = \mathcal{F}^{-1}\Delta_0(\ell)\mathcal{F} =$$

$$= (\mathbf{1}\otimes x^+ - \theta\partial_- \otimes \mathbf{1})^{n+1}(-\mathbf{1}\otimes\partial_+) + (x^+\otimes\mathbf{1} - \mathbf{1}\otimes\theta\partial_-)^{n+1}(-\partial_+\otimes\mathbf{1})$$

and analogously:

$$\Delta_\theta(\ell_n^-) = (-\mathbf{1}\otimes\partial_-)(\mathbf{1}\otimes x^- - \theta\partial_+\otimes\mathbf{1})^{n+1} + (-\partial_-\otimes\mathbf{1})(x^-\otimes\mathbf{1} - \mathbf{1}\otimes\theta\partial_+)^{n+1}$$

The symmetry is a quantum symmetry in the sense that the coproduct is deformed, while the commutation relations of the Lie Coalgebra are the usual ones

Consider the Kaç-Moody algebra in this twisted setting.

$$\left[J^{\pm}(x), J^{\pm}(y)\right]_*, \quad \left[J_+(x), J_-(y)\right]_*$$

Combine fields at different points using the twist, so that, with a slight abuse of notation

$$f(x) * f(y) = m_0 \left[\mathcal{F}(f \otimes g)\right](x,y) = e^{\frac{i}{2}\theta^{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}} f(x) f(y)$$

The deformation in the product does not affect the commutators in the same sector

$$\left[J^{\pm}(x), J^{\pm}(y)\right]_* = -\frac{i}{\pi}\partial_{\mp}\delta(x^{\mp} - y^{\mp})$$

because

$$e^{\frac{i}{2}\partial_x \wedge \partial_y} e^{\mp ikx^{\pm}} e^{\mp ipy^{\pm}} = e^{i\theta(\partial_{x^-}\partial_{y^+} - \partial_{x^+}\partial_{y^-})} e^{\mp ikx^{\pm}} e^{\mp ipy^{\pm}} = e^{\mp ikx^{\pm}} e^{\mp ipy^{\pm}}$$

with

$$\partial_x \wedge \partial_y = \theta^{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}$$

In each chiral sector the symmetry is unchanged. This is to be expected since the $*$-product between two functions of x^+ or x^- alone is the same as the usual product. Likewise the coproduct when acting on pairs of such functions is the same as the undeformed one

The effect of the noncommutativity of the plane is only felt when x^+ and x^- are put together in:

$$\left[J^+(x), J^-(y)\right]_*$$

To calculate it we have to use:

$$e^{\frac{i}{2}\partial_x \wedge \partial_y} e^{-ikx^+} e^{ipy^-} = e^{-i\theta kp} e^{-ikx^+} e^{ipy^-}$$

Using the usual commutation relations for the $a(k)$ we would not obtain commuting currents and the theory would not be conformally invariant.

It is however possible to still obtain the proper algebra with a deformation of the the commutation relations

$$a(p)a(q) = \mathcal{F}^{-1}(p,q)a(q)a(p) + 2p\delta(p+q)$$

where we used the inverse of the twist in momentum space

$$\mathcal{F}^{-1}(p,q) = e^{-\frac{i}{2}p\wedge q} = e^{-i\theta(|p|q-|q|p)}$$

Notice that the action of $\mathcal{F}^{-1}(p,q)$ is always zero when considering the commutator between currents of the same chirality because p and q have the same sign

Thanks to the new twisted commutation relation between the quantum operators $a(p)a(q) = \mathcal{F}^{-1}(p,q)a(q)a(p) + 2p\delta(p+q)$ we obtain the correct algebra.

These twisted, "quantum-plane-like" commutation relations among the a's are not new, they appeared in higher dimensional theories (Balachandran, Mangano, Pinzul, Vaidya)

There a relation of the kind $a(p)a(q) = G(p,q)a(q)a(p)$ was postulated to preserve Lorentz invariance. The use of the twisted coproduct (and the proper limit for $\theta \to 0$) fixes the quantity G to be the twist in dimensions greater than 2.

In two dimensions Lorentz invariance is not enough to fix the commutation relations.

In two-dimensional conformal symmetry, the relations arise naturally upon requesting conformal symmetry

Conclusions and Perspectives

We have constructed a quantum conformal noncommutative field theory

There are several applications of conformal field theory in physics. The theory of strings is the most famous one, but there are also applications in solid state. How is the twist in the commutation relations going to change things?

Likewise the mathematical structure underlying a conformal theory is extremely rich. The whole theory of Vertex Operator Algebras is based on it. The new coproduct is changing the way representations are put together. What structure is the new coproduct putting on the new vertices? on the representations of the algebra? Are they known?

We will probably be busy for a while