Matrix Approximations of Spaces from Noncommutative Geometry

A Physicist's Perspective

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There are several good reasons for which physicists often approximate spaces, or rather functions on a space, with matrices

The most obvious one is the possibility to solve problems numerically

But equally important is the fact that a theory may simply loose its meaning at very short distances (effective theories)

The straightforward *lattice* approximation, whereby we approximate a space with a set of topologically disconnected points, and functions with arrays of numbers multiplied componentwise, looses totally the information of the underlying space, and in particular nearly totally destroys the symmetries of the problem

A lattice approximation for noncommutative spaces is impossible

I will first discuss the noncommutative torus, the archetype of all Noncommutative Geometries

A simple but extremely rich mathematical structure, it has also several physical application.

It is the algebra of elements $a = \sum_{n,m} a_{nm} U_1^n U_2^m$ generated by two unitary generators with the relation

$$U_1 U_2 = e^{2\pi i\theta} U_2 U_1$$

It is possible to study field theories on a NCTorus with the use an integral (trace)

$$\boxed{\int\!\!\!\!\!- \; a := a_{0,0}}$$

and two derivatives

$$\boxed{\partial_i U_j = 2\pi \, \mathrm{i} \, \delta_{ij} U_i} \text{ No sum on } \boxed{i}$$

We can therefore construct, for example, a scalar field theory with action

$$\boxed{S = \int\!\!\!\!\!- \; \mathcal{L}[\Phi, \partial_\mu \Phi]}$$

Let us look at Matrix Approximations to the NCTorus

When the noncommutativity parameter p/q is rational we can try a matrix approximation. That is we substitute the U_i's with their finite dimensional representation (clock and shift) approximations:

$$\mathcal{C}_q = \begin{pmatrix} 1 & & & & \\ & e^{2\pi\,\mathrm{i}\,\frac{p}{q}} & & & \\ & & e^{2\pi\,\mathrm{i}\,\frac{2p}{q}} & & \\ & & & \ddots & \\ & & & & e^{2\pi\,\mathrm{i}\,\frac{(q-1)p}{q}} \end{pmatrix} \;,\quad \mathcal{S}_q = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}$$

$$\mathcal{C}_q \mathcal{S}_q = e^{2\pi \mathrm{i} \frac{p}{q}} \mathcal{S}_q \mathcal{C}_q$$

We can then consider a sequence of algebras with rational values converging the noncommutative torus parameter:

$$\theta_n = \frac{p_n}{q_n} \longrightarrow \theta$$

Thus the noncommutative field theory becomes a matrix model with the projection $$\pi(a) = \sum_{m,r=-\infty}^{\infty} a_{m,r} \ (\mathcal{C}_q)^m \ (\mathcal{S}_q)^r \ .$$

Note however that the noncommutative torus, like the ordinary torus, is not the inductive limit of finite dimensional algebras.

For example the K_1 of an AF algebra is always trivial, the one of the NCTorus is not

Moreover there are no approximations of the derivations on the matrix algebra. So the expression of an action is problematic

It is however possible Pimsner-Voiculescu, LLS to *embed* the non-commutative torus in an AF algebra.

$$A_0 \overset{\rho_1}{\hookrightarrow} A_1 \overset{\rho_2}{\hookrightarrow} A_2 \overset{\rho_3}{\hookrightarrow} \ldots \overset{\rho_n}{\hookrightarrow} A_n \overset{\rho_{n+1}}{\hookrightarrow} \ldots$$

At each level the A_i are sums of the matrix algebras $\mathbf{M}_n(\mathbf{C})$ or their block subalgebras:

$A_1 = \bigoplus_{j=1}^{n_1} \mathbf{M}_{d_j^{(1)}}(\mathbf{C})$ and $A_2 = \bigoplus_{k=1}^{n_2} \mathbf{M}_{d_k^{(2)}}(\mathbf{C})$

but since

$\rho_1(A_1) \subset A_2$

$A_1 \cong \bigoplus_{k=1}^{n_2} \bigoplus_{j=1}^{n_1} N_{kj}\, \mathbf{M}_{d_j^{(1)}}(\mathbf{C})$

ex: $A_1 = \mathbf{M}_3 \oplus \mathbf{M}_2 = \begin{pmatrix} a & \\ & b \end{pmatrix}, A_2 = \mathbf{M}_{13}, \rho(A_1) = \begin{pmatrix} a & & & & \\ & a & & & \\ & & a & & \\ & & & b & \\ & & & & b \end{pmatrix}$

We can find the *Bratteli diagrams*

Considering the expansion of θ as a continued fraction

$$\theta = \lim_n \theta_n = \lim_n \frac{p_n}{q_n}$$

$$\theta_n = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots \; c_{n-1} + \cfrac{1}{c_n}}}}$$

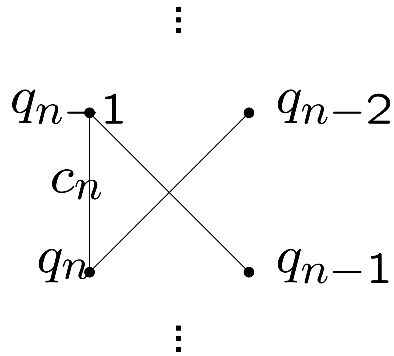
$$p_n = c_n p_{n-1} + p_{n-2} \;,\; p_0 = c_0 \quad , \quad p_1 = c_0 c_1 + 1$$

$$q_n = c_n q_{n-1} + q_{n-2} \;,\; q_0 = 1 \quad , \quad q_1 = c_1$$

$$A_n = \mathbf{M}_{q_n}(\mathbf{C}) \oplus \mathbf{M}_{q_{n-1}}(\mathbf{C})$$

with embeddings $\begin{pmatrix} \mathcal{M} & \\ & \mathcal{N} \end{pmatrix} \xrightarrow{\rho_n} \begin{pmatrix} \left.\begin{matrix} \mathcal{M} & & \\ & \ddots & \\ & & \mathcal{M} \end{matrix}\right\}{\scriptstyle c_n} & & \\ & \mathcal{N} & \\ & & \mathcal{M} \end{pmatrix}$

The corresponding Bratteli diagram is:



Associated we have positive maps $\varphi_n : \mathbf{Z}^2 \to \mathbf{Z}^2$ defined by

$$\begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix} = \varphi_n \begin{pmatrix} q_{n-1} \\ q_{n-2} \end{pmatrix} \qquad , \qquad \varphi_n = \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix} \; .$$

The group $\mathsf{K}_0(A_\infty)$ is the inductive limit

$$\mathsf{K}_0(A_\infty) = \mathbf{Z} + \theta\mathbf{Z}$$

with $\mathsf{K}_0^+(A_\infty) = \left\{ (z,w) \in \mathbf{Z}^2 \mid z + \theta w \geq 0 \right\}$

At a finite level we can define the U's as before

$$U_1^{(n)}U_2^{(n)} = e^{2\pi i p_n/q_n} U_2^{(n)}U_1^{(n)}$$

That we are approximating the torus is given by the relation *(Pimsner–Voiculescu)*

$$\lim_{n\to\infty} \left\| \rho_n \left(U_a^{(n-1)} \oplus U_a^{(n-2)} \right) - U_a^{(n)} \oplus U_a^{(n-1)} \right\|_{A_n} = 0 \quad a = 1,2$$

Note that $U_i^{(\infty)}$ is not a coherent sequence, it is a limit of coherent sequences, therefore

$$A_\theta \subset A_\infty$$

The NCtorus in some sense is on the "boundary" of the algebra

An interesting corollary is the following:

It is a classic result in number theory that two irrational θ, θ' *have the continued fraction expansion which is he same (up to a shift in the indices if and only if they are connected by the* $SL(2, \mathbf{Z})$ *transformation which defines Morita equivalence.*

If A_θ and $A_{\theta'}$ are Morita equivalent, then θ and θ' have the same tail (up to a shift) in the continued fraction expansions, since in a Bratteli diagram what counts is only the infinite tail, the corresponding A_∞ are the same.

Morita equivalent noncommutative tori are subalgebra of the same AF algebra.

We also can define the Hilbert spaces: $\mathcal{H}_n = \mathbf{C}^{q_n} \oplus \mathbf{C}^{q_{n-1}}$ on which the algebra A_n naturally acts.

The embeddings are defined by:

$$\begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} \quad \stackrel{\tilde{\rho}_n}{\longmapsto} \quad \begin{pmatrix} \left.\begin{matrix} \frac{\vec{v}}{\sqrt{1+c_n}} \\ \vdots \\ \frac{\vec{v}}{\sqrt{1+c_n}} \end{matrix}\right\} c_n \\ \\ \vec{w} \\ \\ \frac{\vec{v}}{\sqrt{1+c_n}} \end{pmatrix}$$

Can we use these theories to approximate physical systems?

We can prove that the physical correlation functions can be well approximated (LLS)

Given any two sequences of vectors

$$\psi'_{n-1}, \psi_{n-1} \in \mathcal{H}_{n-1}$$

it results

$$\lim_n \left\langle \psi'_{n-1}\, , \left(U_a^{(n-1)} \oplus U_a^{(n-2)} \right) \psi_{n-1} \right\rangle_{\mathcal{H}_{n-1}}$$
$$- \left\langle \tilde{\rho}_n(\psi'_{n-1})\, , \left(U_a^{(n)} \oplus U_a^{(n-1)} \right) \circ \tilde{\rho}_n(\psi_{n-1}) \right\rangle_{\mathcal{H}_n} = 0$$

We proved (using lots of triangular inequalities) *the limit of the expectation values of all elements of the algebra is well defined in the large N limit.*

This ensures the continuity in θ as well.

Incidentally, a similar construction can be made for the rational case as well

Therefore, while it is true that the algebras for the rational and irrational cases are mathematically very different, physically relevant quantities (such as expectation values etc.) are continuous in θ.

But the derivatives remain problematic, and we can only define an exponentiated version of it

There is a different approximation, based on nonlocal solitons, where the elements of the approximate algebra are matrix valued functions

The noncommutativity of the algebra allows nontrivial projections, selfadjoint elements of the algebra with the property $\boxed{P^2 = P}$
and partial isometries $\boxed{TT^\dagger T = T}$

Projections and partial isometries represent D-brane configurations for type IIA and IIB superstrings respectively. They are local minima of the potential and therefore can be considered solitons of the theory

For our purposes we need a slight generalization of the Powers-Rieffel projection, are ‘ring’-like solitons (Powers-Rieffel solitons) which ‘wrap’ around the torus

Consider a particular sequence of $\theta_n = \dfrac{p_n}{q_n} \to \theta$ and the two sequences of projections

$$\mathsf{P}_n^{11} = V^{-q_{2n-1}}\,\Omega(g_n) + \Omega(f_n) + \Omega(g_n)\,V^{q_{2n-1}}$$

$$\mathsf{P}_n^{11\prime} = U^{q_{2n}}\,\Omega(g_n') + \Omega(f_n') + \Omega(g_n')\,U^{-q_{2n}}$$

with f, g one-dimensional bump functions

We can 'see' these solitons using the fact that the NCtorus is the Weyl quantization of the torus, and use the inverse Wigner map

$$\Omega^{-1}(a)(x,y) = \sum_{m,r=-\infty}^{\infty} a_{m,r} e^{-\pi \mathrm{i}\, m\, r\, \theta} e^{2\pi \mathrm{i}\,(m\, x + r\, y)}$$

The complete projection looks like

As n increases the bumps narrow down and the number of spikes increases

Shifting f and g by $1/q_{2n}$ is possible to obtain other $q_{2n} - 1$ projections P_n^{ii}, exchanging axis one obtains a second set

We can now perform a construction, due to Elliott and Evans, in which we construct a subalgebra of the NCtorus isomorphic to the sum of matrix valued functions on two circles

As $\theta_n \to \theta$ the subalgebra grows and in the limit becomes exactly the algebra of the NCtorus

As the notation suggests the P^{ii} will be the diagonal elements

The nondiagonal elements are built from the partial isometry part of the operators interpolating between the ranges of the projectors: $P^{ii} V P^{jj}$

Since the operator is not selfadjoint its Wigner function is not real, we plot real part and modulus (the imaginary part is qualitatively similar to the real part)

We give the set of projections and isometries, obtained from P or P', the name of tower

The P^{ij} behave as 'matrix units', the basis for a matrix algebra

$$\mathsf{P}^{ij}\,\mathsf{P}^{kl} = \delta_{jk}\,\mathsf{P}^{il}$$

The only problem is in the definition of P^{1q}, which can be done in two different ways, either shifting q times P^{21} and identifying $P^{q+1\ q}$ with P^{1q}, or from $\mathsf{P}^{1q} = \mathsf{P}^{12}\,\mathsf{P}^{23}\ldots\mathsf{P}^{q-1,q}$

The two expressions differ for a partial isometry z

With the identification of $z = e^{2\pi\,\mathrm{i}\,\tau}$ with the exponential of an angle coordinate of a circle, elements of the form $\sum_{i,j=1}^{q}\sum_{k=-\infty}^{\infty} C_{ij}^{k} P^{ij} z^{k}$ close the algebra of matrix valued functions on a circle

After a rotation to make the two towers orthogonal, the same construction can be made in the second tower

Remember that the algebra we have constructed is a subalgebra of the NC-torus at each level of the approximation.

There are two elements of the matrix algebra approximations of U_i in the sense that $\|U_i - \mathsf{U}_i\| \to 0$

Hence we can project all elements of the NCtorus to the sum of matrix valued functions on two circles, making a ‘small’ error. Unlike the usual matrix approximations, the approximation converges strongly

To construct a field theory we need to define derivatives and integral. The integral can be expressed to act on matrix valued functions of τ and τ'

$$\fint \mathsf{a} = \beta \int_0^1 d\tau Tr\, \mathsf{a}(\tau) + \beta' \int_0^1 d\tau' Tr'\, \mathsf{a}'(\tau')$$

with β, β' quantities depending on p, q, p', q'

It is possible to define approximate derivatives $\nabla_1 \mathsf{U} = \mathsf{U}, \nabla_2 \mathsf{V} = \mathsf{V}, \nabla_1 \mathsf{V} = \nabla_2 \mathsf{U} = 0$

The approximation is that they close a Leibnitz rule only in the limit

The expression for the two derivatives is slightly complicated, but it can be given solely in terms of the matrix valued functions, so that it is possible to map the field theory on the noncommutative torus on the action of matrix valued functions of one variable, a Matrix Quantum Mechanics

$$
\begin{aligned}
\boldsymbol{\nabla}_1 \mathsf{a}_n(z,z') \;=\; 2\pi\,\mathrm{i}\,\Bigg[& \sum_{i,j=1}^{q_{2n}} \sum_{k\in\mathbb{Z}} i\,\alpha^{(n)}_{i+\left[\frac{q_{2n}}{2}\right]_0,j;k}\; z^k\, (\mathcal{C}_{q_{2n}})^i\, \Big(\mathcal{S}_{q_{2n}}(z)\Big)^j \\
& \oplus \sum_{i',j'=1}^{q_{2n-1}} \sum_{k'\in\mathbb{Z}} \Big(i' + q_{2n-1}k'\Big)\, {\alpha'}^{(n)}_{i',j'+\left[\frac{q_{2n-1}}{2}\right]_0;k'}\; {z'}^{k'} \Big(\mathcal{S}_{q_{2n-1}}(z')\Big)^{i'}\, (\mathcal{C}_{q_{2n-1}})^{j'} \Bigg] \;,
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\nabla}_2 \mathsf{a}_n(z,z') \;=\; 2\pi\,\mathrm{i}\,\Bigg[& \sum_{i,j=1}^{q_{2n}} \sum_{k\in\mathbb{Z}} (j + q_{2n-1}k)\,\alpha^{(n)}_{i+\left[\frac{q_{2n}}{2}\right]_0,j;k}\; z^k\, (\mathcal{C}_{q_{2n}})^i\, \Big(\mathcal{S}_{q_{2n}}(z)\Big)^j \\
& \oplus \sum_{i',j'=1}^{q_{2n-1}} \sum_{k'\in\mathbb{Z}} j'\, {\alpha'}^{(n)}_{i',j'+\left[\frac{q_{2n-1}}{2}\right]_0;k'}\; {z'}^{k'} \Big(\mathcal{S}_{q_{2n-1}}(z')\Big)^{i'}\, (\mathcal{C}_{q_{2n-1}})^{j'} \Bigg] \;.
\end{aligned}
$$

where z, z' are unitary coordinates on the circles, $\mathcal{C}_{q_{2n}}$ is the usual clock operator defined above, and $\mathcal{S}_{q_{2n-1}}(z)$ is a modification of the shift operator defined as:

$$
\mathcal{S}_q(z) = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ z & & & & 0 \end{pmatrix}
$$

Together they are a basis for the functions on two circles.

Noncommutative Geometry, in its various guises, provides also a way to approximate ordinary spaces using matrices, with the advantage that we can do this preserving the symmetries of the original space

The standard example is the **fuzzy sphere**, loosely speaking, take "coordinates" on 3-d space with

$$[X_i, X_j] = i\frac{r}{\sqrt{N(N+1)}}\varepsilon_{ijk}X_k$$

and the constraint

$$\sum_i X_i^2 = R$$

choosing

$$X_i = \frac{r}{\sqrt{N(N+1)}} L_i$$

with the L's in the $N(N+1)$ representation of $su(2)$

The sphere constraint $X_1^2 + X_2^2 + X_3^2 = r^2$ is just the Casimir of the representation

The algebra is finite dimensional, but rotations act on it, as well as the usual three derivations (which are the X's themselves)

More properly (but it takes longer) one can define coherent states for the representations of $su(2)$, these are finite dimensional matrices, and we can express functions on the sphere in this (truncated) basis

Hence we have a finite dimensional approximation to functions on the sphere

The rotational symmetry of the sphere acts on this algebra by construction, and provides us with a fuzzy Laplacian

$$\nabla^2 f = \sum_i [L_i, [L_i, f]]$$

The eigenvalues of this Laplacian are the same as the usual one on the sphere, truncated to the level $N(N+1)$

The eigenfunctions of this Laplacian are called *fuzzy harmonics*, they are a basis for the matrices $N \times N$

A physicist would say that in the limit the fuzzy sphere converges to the sphere, a mathematician (Rieffel) proves that it is possible to define a distance among quantum Gromov spaces, and to show that with this distance the fuzzy sphere converges to the commutative one.

Similar constructions can be made for all $\mathbf{C}P^n$ spaces, since like the sphere they are coadjoint orbits of groups

To finish however I would like to spend a few words on the fuzzy disc

work in collaboration with P. Vitale & A. Zampini: *Ogni scarrafone è bello 'a mamma sua*

Consider a function on the plane with its Taylor expansion:

$$\varphi(\bar{z}, z) = \sum_{m,n=0}^{\infty} \varphi_{mn}^{\text{Tay}} \bar{z}^m z^n$$

Now "quantize" the plane, using a quantity θ analogous to $\hbar$, and associate to a function the operator

$$\Omega_\theta(\varphi) := \widehat{\varphi} = \sum_{m,n=0}^{\infty} \varphi_{mn}^{\text{Tay}} a^{\dagger m} a^n$$

This is a variant of the Weyl map used to define the Moyal product

Ω_θ has an inverse expressed using *coherent* states:

$$\Omega_\theta^{-1}(\widehat{\varphi}) = \varphi(\bar{z}, z) = \langle z | \, \widehat{\varphi} \, | z \rangle$$

We can express operators with a density matrix notation:

$$\hat{\varphi} = \sum_{m,n=0}^{\infty} \varphi_{mn} |m\rangle \langle n|$$

With $|n\rangle$ eigenvectors of the number operator $\mathrm{N} = a^\dagger a$.

The density matrix basis has a very simple multiplication rule:

$$|m\rangle \langle n| p\rangle \langle q| = \delta_{np} |m\rangle \langle q|$$

The analog of the Taylor expansion in terms of the coefficient φ_{nm} is

$$\varphi(\bar{z}, z) = e^{-\frac{|z|^2}{\theta}} \sum_{m,n=0}^{\infty} \varphi_{mn} \frac{\bar{z}^m z^n}{\sqrt{n! m! \theta^{m+n}}}$$

We have implicitly defined a noncommutative $*$ product on the plane

$$\left(\varphi * \varphi'\right)(\bar{z}, z) = \Omega^{-1}\left(\Omega(\varphi)\,\Omega(\varphi')\right)$$

$$z * \bar{z} - \bar{z} * z = [z, \bar{z}]_* = i\theta$$

In the density matrix basis this product is the usual row by column matrix product

$$\left(\varphi * \varphi'\right)_{mn} = \sum_{k=1}^{\infty} \varphi_{mk}\varphi'_{kn}$$

Also

$$\int dz d\bar{z}\, \varphi(\bar{z}, z) = \frac{1}{2\pi} \operatorname{Tr} \widehat{\varphi} = \frac{1}{\pi} \sum_{n=0}^{\infty} \varphi_{nn}$$

Since we reduced the product to a matrix product this means that we can consider a obtained truncating the expansion to a finite N

It is the algebra obtained with the projector

$$P_\theta^N = \sum_{n=0}^{N} \langle z|\, n\rangle \, \langle n|\, z\rangle = \sum_{n=0}^{N} \frac{r^{2n}}{n!\theta^n} e^{-\frac{r^2}{\theta}}$$

The subalgebra is isomorphic to $N \times N$ matrices

In the combined limit $N \to \infty$ $\theta \to 0$ $N\theta = 1$ the disc becomes sharper

Profile of the spherically symmetric function P_θ^N for the choice $R^2 = N\theta = 1$ and $N = 10, 10^2, 10^3$. As N increases the step becomes sharper.

In the limit we obtain the characteristic function of the disc

Hence, as $\theta = 1/N$ decreases we have a sequence of algebras, and within it a sequence of subalgebras, defined by P_θ^N, made of functions which have mostly support on the unit disc

Rotations are still there, obtained just multiplying the coefficients by a phase.

We can define the matrix equivalent of the derivations as:

$$\partial_z \varphi = \frac{1}{\theta} \langle z| \left[a^\dagger, \Omega(\varphi)\right] |z\rangle$$

$$\partial_{\bar{z}} \varphi = \frac{1}{\theta} \langle z| \left[a, \Omega(\varphi)\right] |z\rangle$$

and project them to obtain fuzzy derivations

The laplacian is a finite operator and we can calculate its eigenvalues, and compare them with the eigenvalues of the Laplacian on the disc

Comparison of the first eigenvalues of the fuzzy Laplacian (circles) with those of the continuum Laplacian (crosses) on the domain of functions with Dirichlet homogeneous boundary conditions. The orders of truncation are $N = 10, 20, 30$

Also in this case one can define fuzzy bessel functions which in the limit converge (inside the disc) to the ordinary Bessel

Comparison of the radial shape for $\Phi^{(N)}_{0,2}(r)$ *with the symbol of the eigenmatrix of the fuzzy Laplacian with respect to the eigenvalue* $\lambda^{(N)}_{0,2}$. *The orders of truncation are* $N = 10, 20, 30$.

One should not try to probe too short distances, there is an Heisenberg uncertainty principle at work, which causes a Ultraviolet-Infrared mixing

Comparison of the radial shape for the $\Phi^{(N)}_{0,10}$. *The orders of truncation are* $N = 30, 35, 40$. *The bump tends to disappear.*

Conclusions

Much of mathematics we would do comes from quantum mechanics and the need to accomodate the geometry of the quantum phase space in the formalism

The original formulation of quantum mechanics was in the form of Matrix Mechanics, which became the theory of operators on Hilbert spaces

Noncommutative Geometry started in mathematics as a filiation of the study of these theory, now however we see that in physics there is an useful point of view to study spaces Matrix Geometry

This is still in need of a more rigorous and deeper understanding. This may produce useful and beautiful mathematics, but will certainly sharpen the tool the physicists use