# A New Matrix Model

# for

# Noncommutative Field Theory

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Suppose we want to discretize the functions on a two dimensional torus.

One option is to consider matrices with entries the values of the functions on a lattice of points.

$$\mathrm{a}(x_1, x_2) \longrightarrow a(i,j)$$

Multiplication is just the product of the single elements

$$(ab)(i,j) = a(i,j)b(i,j)$$

Very little information is carried by this approximation

Moreover we all know the pitfalls of this lattice approximation

We can try a **fuzzy** approximation

*Torus:* $x_1, x_2 \in [0, 1]$

*Functions on a torus:*

$$a(x) = \sum_{mn} a_{mn} e^{2\pi i m x_1} e^{2\pi i n x_2}$$

It is impossible to truncate this sum at a finite level, since the product will produce higher harmonics

Define finite $q$-dimensional clock and shift matrices:

$$U_1 = \begin{pmatrix} 1 & & & & \\ & e^{\frac{2\pi i}{q}} & & & \\ & & e^{2\frac{2\pi i}{q}} & & \\ & & & \ddots & \\ & & & & e^{(q-1)\frac{2\pi i}{q}} \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix} .$$

Now define functions on a fuzzy torus as

$$a = \sum_{m,n=1}^{q} a_{mn} U_1^n U_2^m$$

The sum is finite because $U_1^q = U_2^q = \mathbf{I}$

Now the harmonics retained are finite, the space is *finite* dimensional and the products is consistent at a price

$$U_1 U_2 = e^{\frac{2\pi i}{q}} U_2 U_1$$

# Noncommutative Torus

$$a = \sum_{m,n=-\infty}^{\infty} a_{mn} U_1^n U_2^m$$

$$\boxed{U_1 U_2 = e^{2\pi i\theta} U_2 U_1}$$

## Infinite dimensional algebra

The archetype Noncommutative Compact Geometry

When $\theta$ irrational is not *Morita Equivalent* to any commutative space

The Noncommutative Tori for $\theta$ rational or irrational are "topologically" very different. In general $\theta$ and $\theta' = \frac{a\theta+b}{c\theta+d}$ $ad-bc = 1$ integers give Morita equivalent tori

We can try to approximate the Noncommutative Torus with a sequence of finite matrix algebras generated by shifts and clocks with

$$U_1 = \begin{pmatrix} 1 & & & & \\ & e^{2\pi i \frac{p_n}{q_n}} & & & \\ & & e^{2\,2\pi i \frac{p}{q}} & & \\ & & & \ddots & \\ & & & & e^{(q_n-1)\,2\pi i \frac{p_n}{q_n}} \end{pmatrix}$$

$$\theta_n = \frac{p_n}{q_n} \to \theta$$

a sequence of rationals converging to $\theta$

Is there a sequence of finite dimensional algebras (matrices) ed an embedding map such that the limit (i.e. the completed set of coherent sequences) is the noncommutative torus?

**NO**

The Noncommutative Torus is not an *Approximatively finite algebra*

For example the $K_1$ of an AF algebra is always trivial, the one of the NCTorus is not

This would not be a problem were it not for the fact that we want to approximate this noncommutative geometry (coming from string theory) with a Matrix Algebra

It is however possible Pimsner-Voiculescu, LLS to *embed* the non-commutative torus in an AF algebra.

An AF algebra is a succession of algebras and embeddings:

$$A_0 \overset{\rho_1}{\hookrightarrow} A_1 \overset{\rho_2}{\hookrightarrow} A_2 \overset{\rho_3}{\hookrightarrow} \dots \overset{\rho_n}{\hookrightarrow} A_n \overset{\rho_{n+1}}{\hookrightarrow} \dots$$

At each level the $A_i$ are sums of the matrix algebras $\mathbf{M}_n(\mathbf{C})$ or their block subalgebras:

$A_1 = \bigoplus_{j=1}^{n_1} \mathbf{M}_{d_j^{(1)}}(\mathbf{C})$ and $A_2 = \bigoplus_{k=1}^{n_2} \mathbf{M}_{d_k^{(2)}}(\mathbf{C})$

but since

$\rho_1(A_1) \subset A_2$

$A_1 \cong \bigoplus_{k=1}^{n_2} \bigoplus_{j=1}^{n_1} N_{kj}\, \mathbf{M}_{d_j^{(1)}}(\mathbf{C})$

ex: $A_1 = \mathbf{M}_3 \oplus \mathbf{M}_2 = \begin{pmatrix} a & \\ & b \end{pmatrix}, A_2 = \mathbf{M}_{13}, \rho(A_1) = \begin{pmatrix} a & & & & \\ & a & & & \\ & & a & & \\ & & & b & \\ & & & & b \end{pmatrix}$

Note that it is the embedding which defines the limit: *Bratteli diagrams*

## Embeddings for the noncommutative torus

Any irrational $\theta$ has a unique expansion as a continued fraction:

$$\theta = \lim_n \theta_n = \lim_n \frac{p_n}{q_n}$$

$$\theta_n = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots \; c_{n-1} + \cfrac{1}{c_n}}}}$$

$$\begin{aligned} p_n &= c_n p_{n-1} + p_{n-2} \; , \; p_0 = c_0 \quad , \quad p_1 = c_0 c_1 + 1 \\ q_n &= c_n q_{n-1} + q_{n-2} \; , \; q_0 = 1 \quad , \quad q_1 = c_1 \end{aligned}$$

$$A_n = \mathbf{M}_{q_n}(\mathbf{C}) \oplus \mathbf{M}_{q_{n-1}}(\mathbf{C})$$

with the embeddings:

$$\begin{pmatrix} \mathcal{M} & \\ & \mathcal{N} \end{pmatrix} \overset{\rho_n}{\longmapsto} \begin{pmatrix} \left.\begin{matrix} \mathcal{M} & & \\ & \ddots & \\ & & \mathcal{M} \end{matrix}\right\} c_n & & \\ & \mathcal{N} & \\ & & \mathcal{M} \end{pmatrix}$$

The group $\mathsf{K}_0(A_\infty)$ is the inductive limit

$$\mathsf{K}_0(A_\infty) = \mathbf{Z} + \theta\mathbf{Z}$$

with $\mathsf{K}_0^+(A_\infty) = \left\{(z,w) \in \mathbf{Z}^2 \mid z + \theta w \geq 0\right\}$

At a finite level we can define the $U$'s as before

$$U_1^{(n)} U_2^{(n)} = e^{2\pi i p_n/q_n} U_2^{(n)} U_1^{(n)}$$

That we are approximating the torus is given by the relation *(Pimsner–Voiculescu)*

$$\lim_{n\to\infty} \left\| \rho_n \left( U_a^{(n-1)} \oplus U_a^{(n-2)} \right) - U_a^{(n)} \oplus U_a^{(n-1)} \right\|_{A_n} = 0 \quad a = 1, 2$$

*Note that* $U_i^{(\infty)}$ *is not a coherent sequence, it is a limit of coherent sequences, therefore* $A_\theta \subset A_\infty$

The NCtorus in some sense is on the "boundary" of the algebra

An interesting corollary is the following:

*It is a classic result in number theory that two irrational $\theta, \theta'$ have the continued fraction expansion which is he same (up to a shift in the indices if and only if they are connected by the $SL(2, \mathbf{Z})$ transformation which defines Morita equivalence.*

If $A_\theta$ and $A_{\theta'}$ are Morita equivalent, then $\theta$ and $\theta'$ have the same tail (up to a shift) in the continued fraction expansions, since in a Bratteli diagram what counts is only the infinite tail, the corresponding $A_\infty$ are the same.

**Morita equivalent noncommutative tori**
**are subalgebra of the same AF algebra.**

It is possible to study field theories on a NCTorus with an integral

$$\fint a := a_{0,0}$$

and two derivatives

$$\partial_1 U = 2\pi \mathrm{i}\, U \ , \quad \partial_1 V = 0 \quad ; \quad \partial_2 U = 0 \ , \quad \partial_2 V = 2\pi \mathrm{i}\, V$$

We can therefore construct, for example, a scalar field theory with action

$$S = \fint \mathcal{L}[\Phi, \partial_\mu \Phi]$$

The noncommutative field theory becomes a matrix model

The problem is that there are no approximations of the derivations on the matrix algebra.

I will now present a different approximation, based on nonlocal solitons, whereby the elements of the approximate algebra are matrix valued functions

The noncommutativity of the algebra allows nontrivial projections, selfadjoint elements of the algebra with the property $P^2 = P$
and partial isometries $TT^\dagger T = T$

Projections and partial isometries represent D-brane configurations for type IIA and IIB superstrings respectively. They are local minima of the potential and therefore can be considered solitons of the theory

There are ‘bump’-like localized solitons (Boca projections), the torus equivalent of the gaussian GMS soliton on the Moyal plane. And there are ‘ring’-like solitons (Powers-Rieffel solitons) which ‘wrap’ around the torus

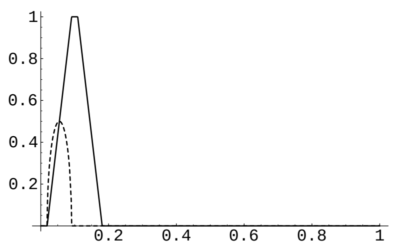
Of course is impossible to ‘see’ these solitons, but we can plot them using the fact that the NCtorus is obtained by the Weyl quantization of the torus, and use the inverse Wigner map $\Omega^{-1}(a)(x,y) = \sum_{m,r=-\infty}^{\infty} a_{m,r} e^{-\pi \mathrm{i}\, m\, r\, \theta} e^{2\pi \mathrm{i}\,(m\, x + r\, y)}$

For our purposes we need a slight generalization of the Powers-Rieffel projection

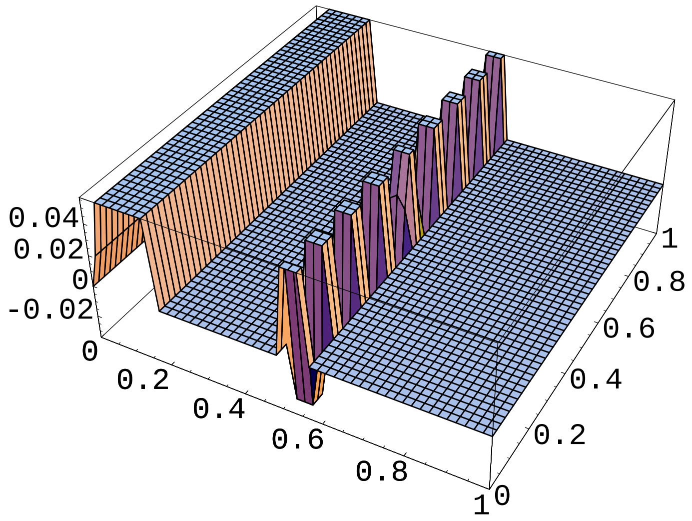
Consider a particular sequence of $\theta_n = \frac{p_n}{q_n} \to \theta$ and the two sequences of projections

$$\mathsf{P}_n^{11} = V^{-q_{2n-1}}\,\Omega(g_n) + \Omega(f_n) + \Omega(g_n)\,V^{q_{2n-1}}$$
$$\mathsf{P}_n^{11\prime} = U^{q_{2n}}\,\Omega(g_n') + \Omega(f_n') + \Omega(g_n')\,U^{-q_{2n}}$$

with $f, g$ one-dimensional bump functions



The complete projection looks like




As $n$ increases the bumps narrow down and the number of spikes increases

$P'$ is obtained exchanging axis

Shifting $f$ and $g$ by $1/q_{2n}$ is possible to obtain other $q_{2n} - 1$ projections $P_n^{ii}$, the same can be done in the second set

We can now perform a construction, due to Elliott and Evans, in which we construct a subalgebra of the NCtorus isomorphic to the sum of matrix valued functions on two circles

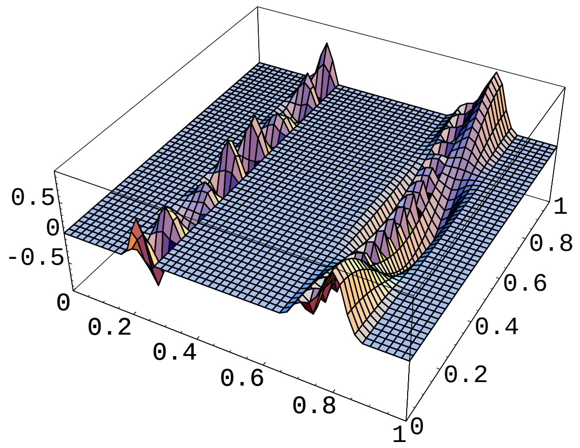
As $\theta_n \longrightarrow \theta$ the subalgebra grows and in the limit becomes exactly the algebra of the NCtorus

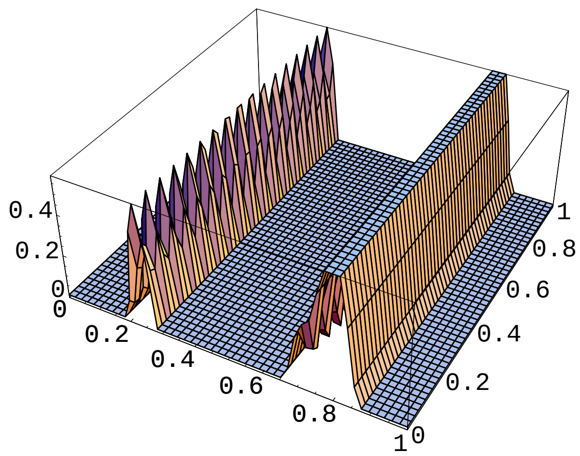
This is not a contradiction since the algebra of functions on a circle is infinite dimensional

As the notation suggests the $P^{ii}$ will be the diagonal elements

The nondiagonal elements are built from the partial isometry part of the operators interpolating between the ranges of the projectors: $P^{ii}VP^{jj}$

Since the operator is not selfadjoint its Wigner function is not real, we plot real part and modulus (the imaginary part is qualitatively similar to the real part)




We give the set of projections and isometries, obtained from $P$ or $P'$, the name of tower

The $P^{ij}$ behave as ‘matrix units’, the basis for a matrix algebra

$$\mathsf{P}^{ij}\,\mathsf{P}^{kl} = \delta_{jk}\,\mathsf{P}^{il}$$

The only problem is in the definition of $P^{1q}$, which can be done in two different ways, either shifting $q$ times $P^{21}$ and identifying $P^{q+1\ q}$ with $P^{1q}$, or from $\mathsf{P}^{1q} = \mathsf{P}^{12}\,\mathsf{P}^{23}\ldots\mathsf{P}^{q-1,q}$

The two expressions differ for a partial isometry $z$

With the identification of $z = e^{2\pi\,\mathrm{i}\,\tau}$ with the exponential of an angle coordinate of a circle, elements of the form $\displaystyle\sum_{i,j=1}^{q}\ \sum_{k=-\infty}^{\infty} C_{ij}^{k} P^{ij} z^{k}$ close the algebra of matrix valued functions on a circle

After a rotation to make the two towers orthogonal, the same construction can be made in the second tower

Remember that the algebra we have constructed is a subalgebra of the NC-torus at each level of the approximation.

There are two elements of the matrix algebra approximations of $U$ and $V$, in the sense that $\|U - \mathsf{U}\| \to 0$ and $\|V - \mathsf{V}\| \to 0$

Hence we can project all elements of the NCtorus to the sum of matrix valued functions on two circles, making a 'small' error. And, unlike the usual matrix approximations, the approximation converges strongly

To construct a field theory we need to define derivatives and integral. The integral can be expressed to act on matrix valued functions of $\tau$ and $\tau'$

$$\oint \mathsf{a} = \beta \int_0^1 d\tau Tr\, \mathsf{a}(\tau) + \beta' \int\limits_0^1 d\tau' Tr'\, \mathsf{a}'(\tau')$$

with $\beta, \beta'$ quantities depending on $p, q, p', q'$

It is possible to define approximate derivatives $\nabla_1 \mathsf{U} = \mathsf{U}, \nabla_2 \mathsf{V} = \mathsf{V}, \nabla_1 \mathsf{V} = \nabla_2 \mathsf{U} = 0$

The approximation is that they close a Leibnitz rule only in the limit

The expression for the two derivatives is slightly complicated, but it can be given solely in terms of the matrix valued functions, so that it is possible to map the field theory on the noncommutative torus on the action of matrix valued functions of one variable, a Matrix Quantum Mechanics

## Three applications

- Perturbative Dynamics of $\boxed{\phi^4}$ theory

  Graphs have ribbon structure. No UV/IR mixing at finite level.

- Tachyon Dynamics

  'time' independent stable configurations correspond to diagonal matrices (all classical vacua commute).

- Yang-Mills Matrix Quantum Mechanics

  Topological Theory. Possibility to sum over the moduli space

We may venture to conclude that this construction shows that branes configurations point to an highly 'delocalized' geometry described better by solitonic vacua that by a deformation of a local theory