Matrix Model

and

the Noncommutative Torus

Fedele Lizzi

With G. Landi & R.J. Szabo

Phys. Lett. **B**578, 449 (2004) `hep-th/0309031`
and `hep-th/0401072` *to appear on ATMP*

Firenze 2004

$$UV = e^{-i2\pi\theta} VU$$

The Noncommutative Torus remains the archetype of all Non-commutative Geometries.

$$a = \sum_{mn} a_{mn} U^m V^n$$

A simple but extremely rich mathematical structure, it has several physical application.

In this talk I will construct an approximation to field theories on the non-commutative torus based on soliton projections and partial isometries which together form a matrix algebra of functions on the sum of two circles.

The approximation is exact in the (inductive) limit of the size of the matrices going to infinity

To study field theories on a NCTorus need an integral (trace)

$$\fint a := a_{0,0}$$

and two derivatives

$$\partial_1 U = 2\pi \mathrm{i}\, U \ , \quad \partial_1 V = 0 \quad ; \quad \partial_2 U = 0 \ , \quad \partial_2 V = 2\pi \mathrm{i}\, V$$

We can therefore construct, say, a scalar field theory

$$S = \fint \mathcal{L}[\Phi, \partial_\mu \Phi] = \fint \partial_\mu \Phi \partial^\mu \Phi + V(\Phi)$$

When $\theta = \frac{p}{q}$ is rational we have a finite dimensional representation of the relation $UV = e^{2\pi \mathrm{i} \theta} VU \equiv \omega VU$ in terms of clock and shift matrices

$$\mathcal{C}_q = \begin{pmatrix} 1 & & & & \\ & \omega^{\frac{p}{q}} & & & \\ & & \omega^{\frac{2p}{q}} & & \\ & & & \ddots & \\ & & & & \omega^{\frac{(q-1)p}{q}} \end{pmatrix} , \quad \mathcal{S}_q = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}$$

Using the fact that any irrational is the limit of rationals (with increasing denominators) we can use clock and shift to define a matrix approximation to the noncommutative Torus.

Define $\lim_n \theta_n = \theta$ with $\theta_n = \frac{p_n}{q_n}$

Thus the noncommutative field theory becomes a matrix model with the projection $\pi_n(a) = \sum_{m,r=-\infty}^{\infty} a_{m,r} \; (\mathcal{C}_{q_n})^m \; (\mathcal{S}_{q_n})^r$.

The error is due to the fact that $(\mathcal{C}_{q_n})^{q_n} = (\mathcal{S}_{q_n})^{q_n} = 1$

So for n and q_n large one can hope the the approximation is good for "low energy" fields, for which higher Fourier modes do not dominate.

However the noncommutative torus, like the ordinary one, is not the inductive limit of finite dimensional algebras.

Moreover there are no approximations of the derivations on the matrix algebra.

I will present a different approximation, based projections (solitons). The elements of the approximate algebra will be matrix valued functions on two circles

The noncommutativity of the algebra allows nontrivial projections, selfadjoint elements of the algebra with the property $P^2 = P$
and partial isometries $TT^\dagger T = T$

Projections and partial isometries represent D-brane configurations for type IIA and IIB superstrings respectively. They are local minima of the potential and therefore can be considered solitons of the theory

In the commutative case the projections' would be characteristic functions, which not being continuous, do not belong to the algebra. They are however the basis for a lattice approximation. No sequence of functions on a lattice converges uniformly to the algebra of continuous functions.

In the noncommutative case there are 'bump'-like localized solitons (Boca projections), which resemble the charateristic functions. They are the torus equivalent of the gaussian GMS soliton on the Moyal plane.

There are also 'ring'-like solitons (Powers-Rieffel projections) which 'wrap' around the torus.

Of course is impossible to 'see' these solitons, but we can plot them using the fact that the NCtorus is obtained by the Weyl quantization of the torus, and use the inverse Wigner map

$$\Omega^{-1}(a)(x,y) = \sum_{m,r=-\infty}^{\infty} a_{m,r} e^{-\pi \mathrm{i}\, m\, r\, \theta} e^{2\pi \mathrm{i}\,(m\,x + r\,y)}$$

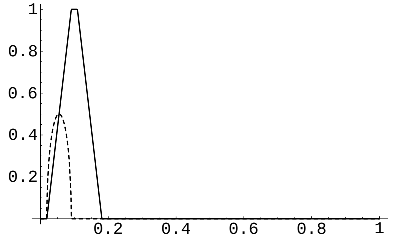
For our purposes we need a family of generalized Powers-Rieffel projection

Consider a particular sequence of $\theta_n = \dfrac{p_n}{q_n} \to \theta$ and the two sequences of projections

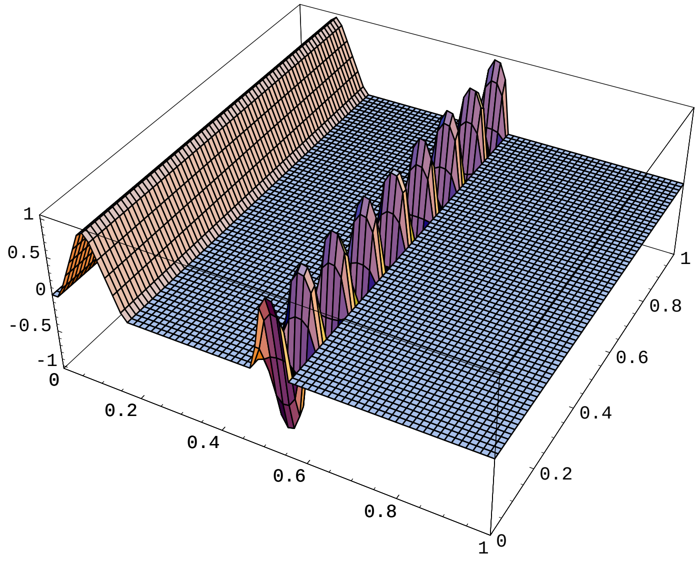
$$\mathsf{P}^{11}_n = V^{-q_{2n-1}}\,\Omega(g_n) + \Omega(f_n) + \Omega(g_n)\,V^{q_{2n-1}}$$

$$\mathsf{P}^{11'}_n = U^{q_{2n}}\,\Omega(g'_n) + \Omega(f'_n) + \Omega(g'_n)\,U^{-q_{2n}}$$

with f, g one-dimensional bump functions



The complete projection looks like



As n increases the bumps narrow down and the number of spikes increases

P' is obtained exchanging axis.

Shifting f and g by $1/q_{2n}$ is possible to obtain other $q_{2n} - 1$ projections P_n^{ii}, the same can be done in the second set

We can now perform a construction, due to Elliott and Evans, in which we construct a subalgebra of the NCtorus isomorphic to the sum of matrix valued functions on two circles

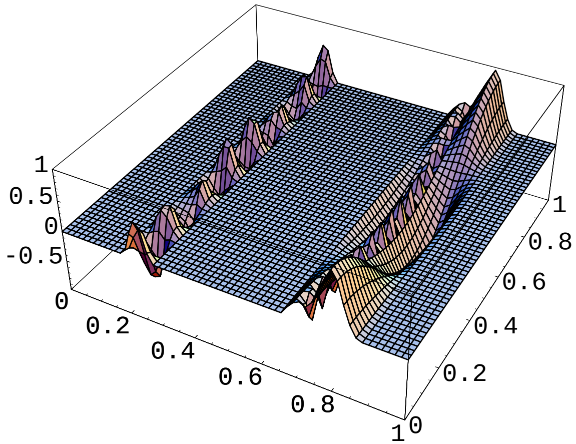
As $\theta_n \longrightarrow \theta$ the subalgebra grows and in the limit becomes exactly the algebra of the NCtorus

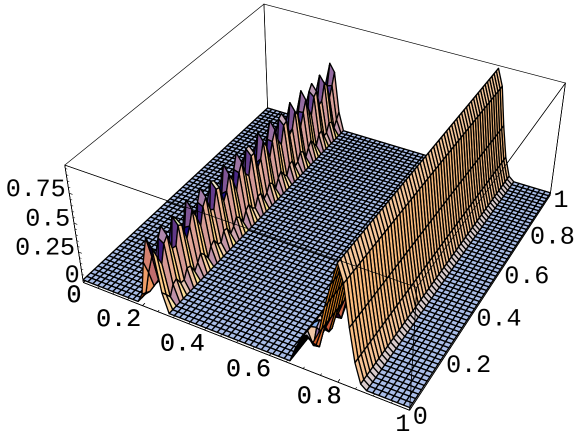
This is not a contradiction since the algebra of functions on a circle is infinite dimensional

As the notation suggests the P^{ii} will be the diagonal elements

The nondiagonal elements are built from the partial isometry part of the operators interpolating between the ranges of the projectors: $P^{ii} V P^{jj}$

Since the operator is not selfadjoint its Wigner function is not real, we plot real part and modulus (the imaginary part is qualitatively similar to the real part)





The set of projections and isometries, obtained from P (or P'), is called tower

The P^{ij} behave as 'matrix units', the basis for a matrix algebra

$$\mathsf{P}^{ij}\,\mathsf{P}^{kl} = \delta_{jk}\,\mathsf{P}^{il}$$

The only problem is in the definition of P^{1q}, which can be done in two different ways, either shifting q times P^{21} and identifying $P^{q+1\ q}$ with P^{1q}, or from $\mathsf{P}^{1q} = \mathsf{P}^{12}\,\mathsf{P}^{23}\cdots\mathsf{P}^{q-1,q}$

The two expressions differ for a partial isometry z

With the identification of $z = e^{2\pi\,\mathrm{i}\,\tau}$ with the exponential of an angle coordinate of a circle, elements of the form $\displaystyle\sum_{i,j=1}^{q}\ \sum_{k=-\infty}^{\infty} C_{ij}^{k} P^{ij} z^{k}$ close the algebra of matrix valued functions on a circle

After a rotation to make the two towers orthogonal, the same construction can be made in the second tower

The algebra we have constructed is a subalgebra of the NCtorus at each level of the approximation.

And at the same time is isomorphic to the algebra of matrix valued functions on two circles

There are two elements of the matrix algebra approximations of U and V, in the sense that $\|U - \mathsf{U}\| \to 0$ and $\|V - \mathsf{V}\| \to 0$

Hence we can project all elements of the NCtorus to the sum of matrix valued functions on two circles, making a ‘small’ error. And, unlike the usual matrix approximations, the approximation converges strongly

To construct a field theory we need to define derivatives and integral. The integral can be expressed to act on matrix valued functions of τ and τ'

$$\fint \mathsf{a} = \beta \int_0^1 d\tau Tr\, \mathsf{a}(\tau) + \beta' \int\limits_0^1 d\tau' Tr'\, \mathsf{a}'(\tau')$$

with β, β' quantities depending on p, q, p', q'

It is possible to define approximate derivatives $\nabla_1 \mathsf{U} = \mathsf{U}, \nabla_2 \mathsf{V} = \mathsf{V}, \nabla_1 \mathsf{V} = \nabla_2 \mathsf{U} = 0$

The approximation is that they close a Leibnitz rule only in the limit

The expression for the two derivatives is slightly complicated

$$\begin{aligned}
\nabla_1 \mathsf{a}_n(\tau,\tau') &= \Sigma \mathsf{a}_n(\tau) \oplus \left(q_{2n-1}\, \dot{\mathsf{a}}'_n(\tau') + \left[\Xi', \mathsf{a}'_n(\tau')\right]\right) , \\
\nabla_2 \mathsf{a}_n(\tau,\tau') &= \left(q_{2n}\, \dot{\mathsf{a}}_n(\tau) + \left[\Xi, \mathsf{a}_n(\tau)\right]\right) \oplus \Sigma' \mathsf{a}'_n(\tau') ,
\end{aligned}$$

where $\dot{\mathsf{a}}_n(\tau) := \mathrm{d}\mathsf{a}_n(\tau)/\mathrm{d}\tau$ and $\dot{\mathsf{a}}'_n(\tau') := \mathrm{d}\mathsf{a}'_n(\tau')/\mathrm{d}\tau'$
$\Xi_{ij} := 2\pi\,\mathrm{i}\, j\ \delta_{ij}$

$\Sigma \mathsf{a}_n(\tau)_{ij} = \sum_{s,t} \Sigma(\tau)_{ij,st}\, \mathsf{a}_n(\tau)_{st}$, with

$$\Sigma(\tau)_{ij,st} = \Sigma_{is} \times \begin{cases} \delta_{t,s+j-i} & i<j \quad 1 \le s \le q_{2n}+i-j \\ \delta_{t,s+j-i-q_{2n}}\ e^{2\pi\,\mathrm{i}\,\tau} & i<j \quad q_{2n}+i-j+1 \le s \le q_{2n} \\ \delta_{t,s+i-j} & j \le i \quad 1 \le s \le q_{2n}+j-i \\ \delta_{t,s+i-j-q_{2n}}\ e^{-2\pi\,\mathrm{i}\,\tau} & j<i \quad q_{2n}+j-i+1 \le s \le q_{2n} \end{cases}$$

What is important is that it can be given solely in terms of the matrix valued functions, so that it is possible to map the field theory on the noncommutative torus on the action of matrix valued functions of one variable, a Matrix Quantum Mechanics

Three applications (which I have no time to discuss)

- Perturbative Dynamics of ϕ^4 theory

  Graphs have ribbon structure. No UV/IR mixing at finite level.

- Tachyon Dynamics

  'time' independent stable configurations correspond to diagonal matrices (all classical vacua commute).

- Yang-Mills Matrix Quantum Mechanics

  Topological Theory. Possibility to sum over the moduli space

We may venture to conclude that this construction shows that branes configurations point to an highly 'delocalized' geometry described better by solitonic vacua that by a deformation of a local theory