Physical Applications of Noncommutative Geometry

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It is useful to start with what Noncommutative Geometry is not

- Noncommutative Geometry is not the theory of a spacetime in which $[x_i, x_j] = i\theta_{ij}$

- Noncommutative Geometry is not something discovered in string-brane theory

- Noncommutative Geometry is not a matrix model

Noncommutative Geometry is also this, but it is reductive to think of it just in terms of one of these way.

The starting point (at least in my opinion) is that all topological properties of a topological space are encoded in the algebra of continuous functions on it

Any commutative C^*-algebra is the algebra of continuous functions from a topological space into complex numbers. The space (with its topology) can be reconstructed as the space of characters (*the spectrum)* of the algebra.

The set of points of the spectrum of the (commutative) algebra $\mathcal{A}$ is given by the characters, seen as evaluation maps:

$$x : f \in \mathcal{A} \to \mathbf{C} \mid x(f) = f(x)$$

All C^*-algebra can be represented as bounded operators on an Hilbert space. Other geometrical properties of the space (differential, metric, homological) can be recovered with presence of another operator, some sort of generalized derivative operator or laplacian.

For example, given the (generalized) Dirac Operator, self-adjoint, not necessarily bounded operator D with compact resolvent, such that $[D, f]$ is bounded for f in a dense subset of the algebra.it is possible to define the distance between two points (pure states):

$$d(x, y) = \sup_{||D,f|| \leq 1} |x(f) - y(f)| = \sup_{||D,f|| \leq 1} |f(x) - f(y)|$$

With the operator D it is possible to represent differential forms (connections, potentials) as operators on the Hilbert space:

$$\pi(f\, dg) = f[D, g]$$

for higher forms quotients are necessary (junk forms)

As physicists we may say that (complex valued) fields and derivatives of a second quantized theory are sufficient to recover the geometry of the space.

The algebraic point of view is equivalent to the ordinary pointlike one, and is also possible to set up a dictionary

topological space	commutative C^* -algebra
points	space of characters
integral	trace
forms df	Operators $[D, f]$
gauge group	unitary elements of the algebra

The setting is such that is can immediately be generalized to noncommutative algebras leading to a Noncommutative Geometry. We do not have an unambiguous characterization of points, but the setting is otherwise unaltered.

But why do we need a Noncommutative Geometry?

We already have a Noncommutative Geometry: Quantum Mechanics

The phase space of quantum systems, original Noncommutative Geometry

At scales of the order of $\hbar$ the geometrical structure of the pahse space becomes noncommutative. The description can only happen as a noncommutative algebra of operators.

There are several indications (black hole physics, strings, loop quantum gravity etc.) that at scales of the Planck length ordinary geometry may not be adequate

Still we need to find, at some scales, ordinary geometry. A problem analogous to the classical limit of quantum mechanics.

We maybe interested in a deformation of a commutative algebra (space), governed by a small parameter θ:

the $\star$ or Grönewold-Moyal product:

$$(f \star g)(x) = e^{i\theta\varepsilon^{ij}\partial_{\xi_i}\partial_{\eta_j}} f(\xi)g(\eta)\Big|_{\xi=\eta=x} = fg + i\theta\{f, g\} + O(\theta^2)$$

$\{f, g\}$ the Poisson Bracket. This is a way to quantize a phase space, called deformation quantization

But let me honest, why am I here today?

Emergence of a noncommutative geometry in string theory

Seiberg and Witten 1998

In the presence of a nontrivial torsion background B the action of an open string is:

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} g_{ij} \partial_a x^i \partial^a x^j - \frac{i}{2} \int_{\partial\Sigma} B_{ij} x^i \partial_t x^j$$

and the propagator between two x's is:

$$\left\langle x^i(\tau) x^j(\tau') \right\rangle = -\alpha' G^{ij} \log(\tau - \tau') + \frac{i}{2} \theta^{ij} \Theta(\tau - \tau')$$

with G and θ symmetric and antisymmetric parts of $(g + 2\pi\alpha B)^{-1}$ respectively.

Θ the step function.

The leading short distance singularity for two fundamental (tachyon) vertex operators $V_p(\tau) = e^{ip\cdot x(\tau)}$ is $V_p(\tau)\cdot V_q(\tau') = (\tau-\tau')^{2\alpha' p^i G_{ij} p^j}\ e^{-\frac{1}{2}\theta^{ij}p_i q_j} V_{p+q}$

In the limit $\alpha' \to 0$ these are the generators of a *deformed* $\star$ algebra.

The full noncommutative geometry of a interacting strings would be described by the enormous algebra of vertex operators

Vertex operators algebra (seen as generalizations of Lie algebras) are a very important branch of mathematics which connects group theory, conformal field theory, harmonic complex analysis, numenr theory . . .

In this it is possible to see naturally" all simmetries T-duality, reparametrization invariance of the target space, world shhet conformal invariance, as gauge transformations

String theory has prompted an intense study of field theories in which the fields are multiplied with the $\star$ product, or

Noncommutative Field Theories

Most of the activity has been for the simplest Moyal $\star$-product, but some work has been done also for κ Minkowski space, $[x_i, x_0] = \kappa^{-1} x_i, / [x_i, x_j] = 0$

The most striking effect in Noncommutative Field Theory is the Ultraviolet-Infrared mixing.

Since the theory has a scale (loosely identifiable with Planck Length), its ultraviolet properties are modified, and some previously UV divergent diagrams become convergent. But because of the presence of nonorientable diagrams of new IR divergence appear.

A Noncommutative Geometry is a an algebra which can always be represented as operators on an Hilbert space. And, as our students teach us, operators are but infinite matrices. And infinite matrices can be truncated, hoping that the finite model retains some of the result of the original one.

In a nutshell this the connection between NCG, Matrix Models, and fuzzy spaces.

fuzzy sphere approssimazioni finite

The fuzzy sphere is defined by $x_1^2 + x_2^2 + x_3^2 = 1$. Imposing

$$[x_i, x_j] = \frac{i}{N(N+1)} \varepsilon_{ijk} x_k$$

Thus obtain a noncommutative space. With $x_i \propto L_i$ in the representation $N \times N$ we obtain an approximation of the algebra of the sphere.

prospettive e lista delle cose di cui non ho parlato