The original title of my talk was:

NONCOMMUTATIVE GEOMETRY

but upon receiving the programme I discovered they wanted me to talk of:

NONCOMMUNICATIVE GEOMETRY

I don't know what the latter is, so I will have to make it up.

We live in the age of communications, so the points of spacetime communicate with each other via fields.

But just as in the case of modern days internet. Too much communication is not necessarily a good thing!

Points that are too close to each other communicate too much, and hence they blow up, so we have to impose a cutoff

In a *Noncommunicative Geometry* points don't talk so much each other. The fields are finite, and therefore they cannot give infinities

A first example of such Noncommunicative Geometry is a Lattice

The algebra of fields φ_I defined on a lattice is just a collection of points with pointwise multiplication

$$(\varphi\psi)_I = \varphi_I \psi_I$$

Lattices have very few degrees of freedom

And they have very few symmetries

The former is good (renormalization, calculations)
The latter is bad, symmetries (and broken symmetries) are essential features of a physical theory. We want to keep the symmetries of spacetime.

Actually it is not true that on a lattice points do not talk to each other. They do via derivatives, Laplacians, Dirac operators .... We need to take this into account.

For example the derivative on the direction $\vec{k}$ on a lattice with spacing a is: $\boxed{(\nabla_{\vec{k}}\varphi)_I = \frac{\varphi_{I+\vec{k}} - \varphi_I}{a}}$

So we want to find a way to obtain finite space which retain as many as possible of the symmetries of the continuum space

I will just present some examples. I do not claim that they are a fair rendition of spacetime as a quantum space. But they indicate at least possibility that at short distances symmetries can be retained and degrees of freedom cut

Suppose we want to discretize the functions on a two dimensional torus.

One option is to consider matrices with entries the values of the functions on a lattice of points.

Multiplication is just the product of the single elements

Translational symmetry is lost, apart from a small subgroup

We can try a **fuzzy** approximation

Torus: $x_1, x_2 \in [0,1]$

Functions on a torus:

$\varphi(x) = \sum_{mn} \varphi_{mn} e^{2\pi i m x_1} e^{2\pi i n x_2}$

It is impossible to truncate this sum at a finite level, since the product will produce higher harmonics

Define finite N-dimensional clock and shift matrices:

$$U_1 = \begin{pmatrix} 1 & & & & \\ & e^{\frac{2\pi i}{N}} & & & \\ & & e^{2\frac{2\pi i}{N}} & & \\ & & & \ddots & \\ & & & & e^{(N-1)\frac{2\pi i}{N}} \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}.$$

$$\varphi = \sum_{m,n=1}^{N} \varphi_{mn} U_1^n U_2^m$$

The sum is finite because $U_1^N = U_2^N = \mathbf{I}$

Now the harmonics retained are finite, the space is *finite* dimensional and the products is consistent at a price $U_1 U_2 = e^{\frac{2\pi i}{N}} U_2 U_1$

After all Noncommunicative Geometry is Noncommutative Geometry

So I can go on with my seminar after all!

On a fuzzy Torus there is the $U(1) \times U(1)$ translations group unchanged.

For example translation of angles α_k in the direction k are: $\varphi_{mn} \to e^{i(m\alpha_1 + n\alpha_2)}$

There are the two derivatives:

$$\nabla_1 \varphi = \sum_{m,n=1}^{N} m \varphi_{mn} U_1^n U_2^m$$

The spectrum of the Laplacian is the same as in the communicative case, only truncated at level N

If N large enough, and hence the noncommutativity small enough, the two spaces probed by a low energy theory would look the same

Another famous example is the fuzzy sphere

$$[X_i, X_j] = i\frac{r}{\sqrt{N(N+1)}}\varepsilon_{ijk}X_k$$

choosing

$$X_i = \frac{r}{\sqrt{N(N+1)}}L_i$$

with the L's in the $N(N+1)$ representation of $su(2)$

The sphere constraint $X_1^2 + X_2^2 + X_3^2 = r^2$ is just the Casimir of the representation

The algebra is finite dimensional, but rotations act on it, as well as the usual three derivations (which are the X's themselves)

Also in this case the Laplacian is the same as the one on the sphere, but is truncated

Let me now introduce the fuzzy disc

work in collaboration with P. Vitale & A. Zampini

Consider a function on the plane with its Taylor expansion:

$$\varphi(\bar{z}, z) = \sum_{m,n=0}^{\infty} \varphi_{mn}^{\mathsf{Tay}} \bar{z}^m z^n$$

Now "quantize" the plane, using a quantity θ analogous to $\hbar$, and associate to a function the operator

$$\Omega_\theta(\varphi) := \hat{\varphi} = \sum_{m,n=0}^{\infty} \varphi_{mn}^{\mathsf{Tay}} a^{\dagger m} a^n$$

This is a variant of the Weyl map used to define the Moyal product

Ω_θ has an inverse expressed using *coherent* states:

$$\Omega_\theta^{-1}(\hat{\varphi}) = \varphi(\bar{z}, z) = \langle z | \, \hat{\varphi} \, | z \rangle$$

We can express operators with a density matrix notation:

$$\hat{\varphi} = \sum_{m,n=0}^{\infty} \varphi_{mn} |m\rangle \langle n|$$

With $|n\rangle$ eigenvectors of the number operator $\mathrm{N} = a^\dagger a$.

The density matrix basis has a very simple multiplication rule:

$$|m\rangle \langle n| p\rangle \langle q| = \delta_{np} |m\rangle \langle q|$$

The analog of the Taylor expansion in terms of the coefficient φ_{nm} is

$$\varphi(\bar{z}, z) = e^{-\frac{|z|^2}{\theta}} \sum_{m,n=0}^{\infty} \varphi_{mn} \frac{\bar{z}^m z^n}{\sqrt{n!m!\theta^{m+n}}}$$

We have implicitly defined a noncommutative $*$ product on the plane

$$\left(\varphi * \varphi'\right)(\bar{z}, z) = \Omega^{-1}\left(\Omega(\varphi)\,\Omega(\varphi')\right)$$

$$z * \bar{z} - \bar{z} * z = [z, \bar{z}]_* = i\theta$$

In the density matrix basis this product is the usual row by column matrix product

$$\left(\varphi * \varphi'\right)_{mn} = \sum_{k=1}^{\infty} \varphi_{mk}\varphi'_{kn}$$

Also

$$\int dz d\bar{z}\, \varphi(\bar{z}, z) = \frac{1}{2\pi} \operatorname{Tr} \widehat{\varphi} = \frac{1}{\pi} \sum_{n=0}^{\infty} \varphi_{nn}$$

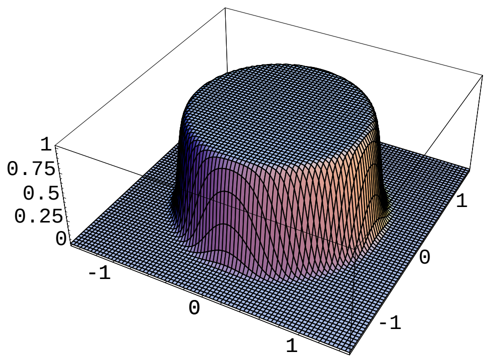
Since we reduced the product to a matrix product this means that we can consider a obtained truncating the expansion to a finite N

It is the algebra obtained with the projector

$$P_\theta^N = \sum_{n=0}^{N} \langle z|\, n\rangle \, \langle n|\, z\rangle = \sum_{n=0}^{N} \frac{r^{2n}}{n!\theta^n} e^{-\frac{r^2}{\theta}}$$

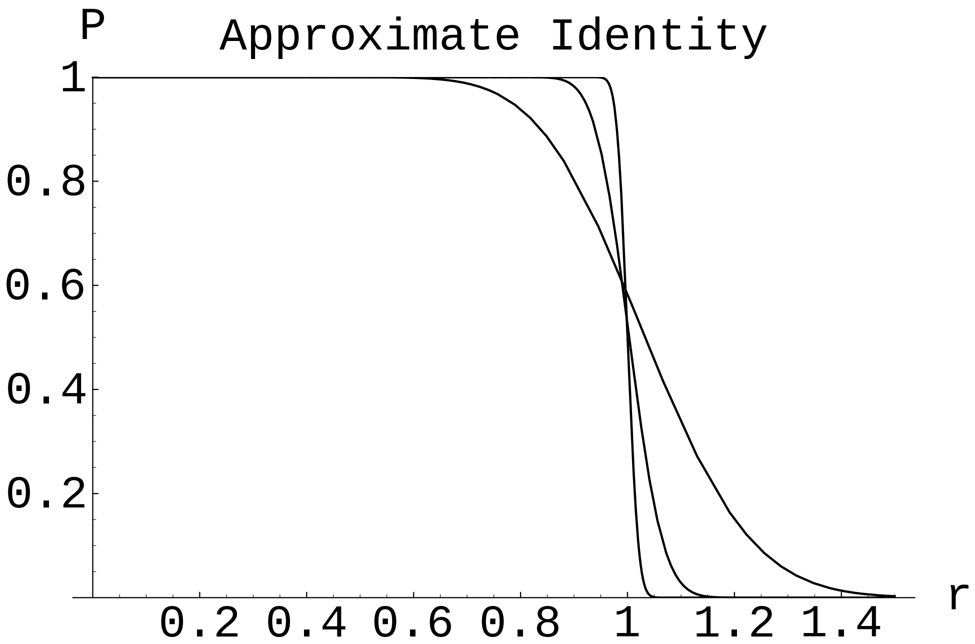
The subalgebra is clearly isomorphic $N \times N$ matrices

What sort of functions are there in this algebra?



The function P_θ^N for $N = 10^2$, $\theta = 1/N$.

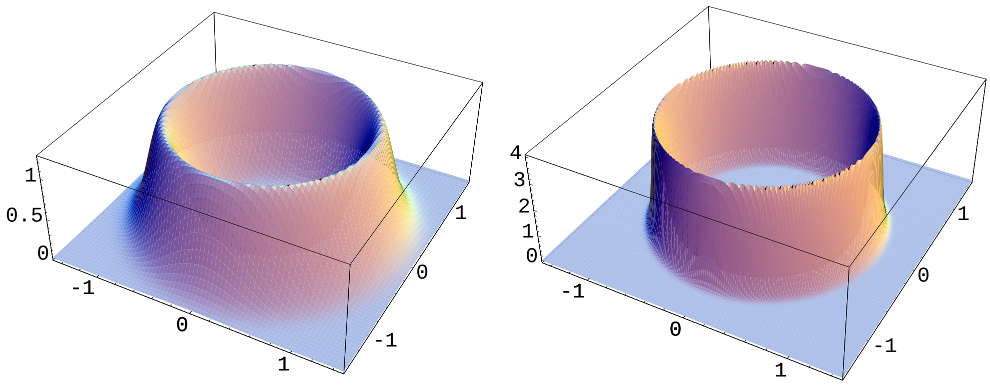
The disc becomes sharper as N increases, keeping $\boxed{R = \theta}$ fixed



Profile of the spherically symmetric function P_θ^N for the choice $R^2 = N\theta = 1$ and $N = 10, 10^2, 10^3$. As N increases the step becomes sharper.

The subalgebra defined by $\boxed{P_\theta^N}$ is therefore made of functions which have mostly support on the disc of radius $\boxed{R = 1/\theta}$. More sharply defined as N increases

We also have edge states



The edge states $\langle z|N\rangle\langle N|z\rangle$ for $N = 10$ and $N = 100$.

Rotations are still there, obtained just multiplying the coefficients by a phase.

Still I need more to convince you that this matrix algebra has to do with the disc.

Derivatives and Laplacians

The starting point to define the matrix equivalent of the derivations is:

$$\partial_z\varphi = \frac{1}{\theta}\langle z|\,[a^\dagger,\Omega(\varphi)]\,|z\rangle$$

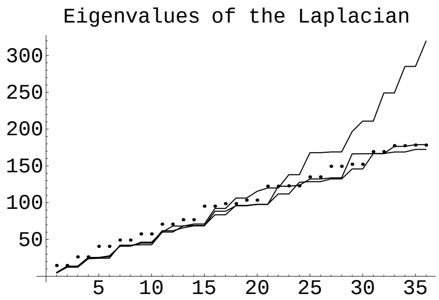
$$\partial_{\bar{z}}\varphi = \frac{1}{\theta}\langle z|\,[a,\Omega(\varphi)]\,|z\rangle$$

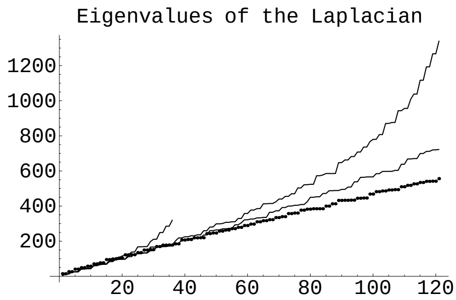
Note that $\partial_z(P_\theta^N * \varphi * P_\theta^N) \neq \Pi_\theta^N * (\partial_z\varphi)P_\theta^N$ but the latter is simpler to implement for matrices

Since $\boxed{a}$ and $\boxed{a^\dagger}$ are still infinite matrices, $\boxed{\widehat{\partial}_z \widehat{\varphi}}$ and $\boxed{\widehat{\partial}_{\bar{z}} \widehat{\varphi}}$ are $\boxed{N+1 \times N+1}$ matrices

The fact that functions and "form" live in different space is standard in NCG

The laplacian is a finite operator and we can calculate its eigenvalues, and compare them with the eigenvalues of the Laplacian on the disc





The first eigenvalues of the laplacian on the disc (dots) and the fuzzy laplacian (solid lines) for $N = 5, 10, 15$. *The lines corresponding to the three cases can be distinguished because the agreement with the exact case improves as* N *grows. In the figure on the right the curve which interrupts is the one corresponding to* $N = 5$, *for which there are only 36 eigenvalues.*

Massless Scalar Field Theory

The setting is ready to solve nonperturbatively theories with a path integral formalism

To a an action of the type:

$$S = \int d^2 z \varphi \nabla^2 \varphi + \frac{m^2}{2} \varphi^2$$

We associate the fuzzy action

$$S_\theta^2 = \frac{1}{\pi} \operatorname{Tr} \widehat{\varphi} \widehat{\nabla}^2 \widehat{\varphi} \frac{m^2}{2} \widehat{\varphi}^2 + V(\widehat{\varphi})$$

A field theory can be solved with a path integral Montecarlo or other techniques

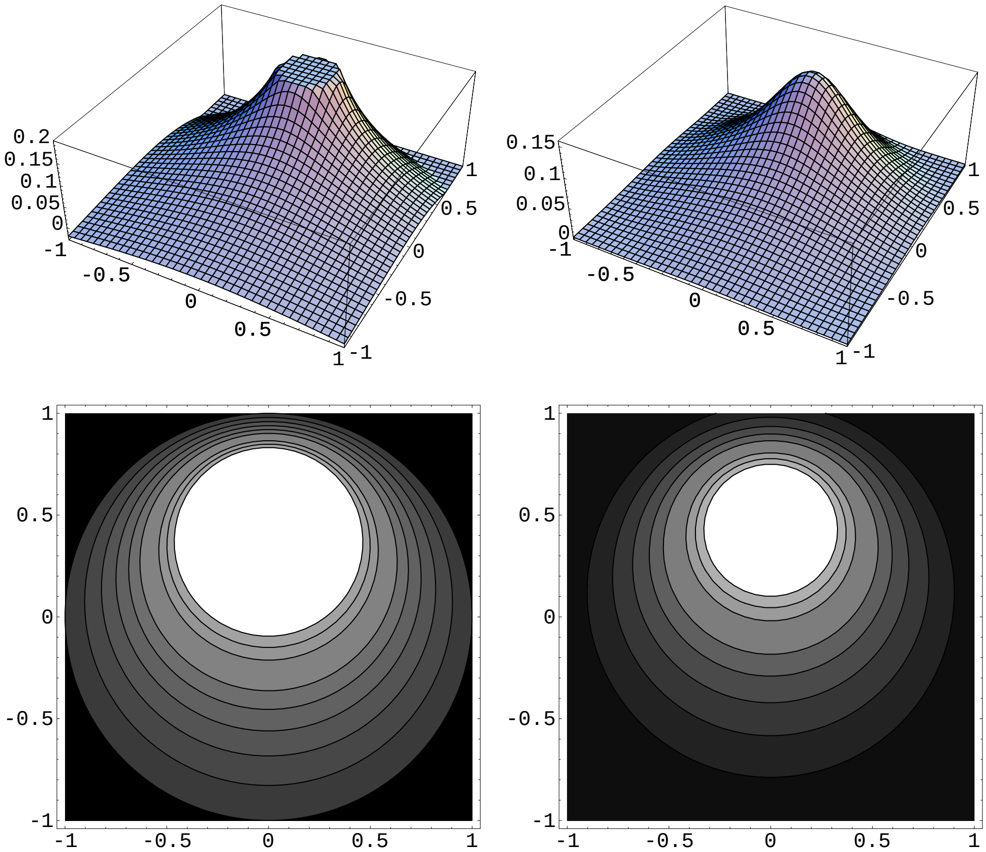
A particularly simple case is when the theory is free and massless, then

$$\langle\varphi(\bar{z},z)\varphi(\bar{z}',z')\rangle = G(z,z')$$

The path integral gives just the inverse of the Laplacian

$$G_\theta^{(N)}(z,z') = \sum_{m,n,p,q=1}^{N} \frac{e^{-\frac{|z|^2+|z'|^2}{\theta}}(\widehat{\nabla}^{-2})_{mnpq}\bar{z}^p z^q z'^m \bar{z}'^n}{\sqrt{p!q!m!n!\theta^{m+n+p+q}}}$$

We can therefore compare the exact Green's function on the Disc with our approximation



Comparison of the 3D and contour plot of $G(z, z')$ for

$z = 0 + 1/2i$ as a function of z'

The fuzzy case (right) is with $N = 15$

Conclusions

There are indications (for example from string theory) that at very short distances the structure of spacetime may be noncommutative

If we are willing to give up commutativity then we open the possibility that spacetime may be represented by matrices rather than continuous functions

All this still retaining the fundamental symmetries of spacetime