Fuzzy Discretization

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Lecce 2003

Usually the way to discretize a system (for the s
differentail equation, the study of a filed theory etc.)

The algebra of fields φ_I defined on a lattice is just
of points with pointwise multiplication

$$(\varphi\psi)_I = \varphi_I \psi_I$$

Lattices have very few degrees of freedom

And they have very few symmetries

The former is good (renormalization, possibility to do calculat
The latter is bad, symmetries (and broken symmetries) are ess
of a physical theory. We want to keep the symmetries of spac

A lattice is not just a collection of points, there are a
among those: derivatives, Laplacians, Dirac operato
define which type of lattice we are dealing with.

For example the derivative on the direction $\vec{k}$ on a lattice with spacing a is

$$(\nabla_{\vec{k}}\varphi)_I = \frac{\varphi_{I+\vec{k}} - \varphi_I}{a}$$

So we want to find a way to obtain a finite space
as many as possible of the symmetries of the contin

Here I will present some examples of a different *no*
discretization which, while cutting the degrees of fre
theory, retain all basic symmetries of it.

Suppose we want to discretize the functions on a two

torus.

One option is to consider matrices with entries the v

functions on a lattice of points.

Multiplication is just the product of the single elements

Translational symmetry is lost, apart from a small s

We can try a **fuzzy** approximation

Torus: $x_1, x_2 \in [0,1]$

Functions on a torus:

$$\varphi(x) = \sum_{mn} \varphi_{mn}\, e^{2\pi i m x_1}\, e^{2\pi i n x_2}$$

It is impossible to truncate this sum at a finite lev
product will produce higher Fourier modes

Define finite N-dimensional clock and shift matrices:

$$U_1 = \begin{pmatrix} 1 & & & & \\ & e^{\frac{2\pi i}{N}} & & & \\ & & e^{2\frac{2\pi i}{N}} & & \\ & & & \ddots & \\ & & & & e^{(N-1)\frac{2\pi i}{N}} \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix} .$$

$$\varphi = \sum_{m,n=1}^{N} \varphi_{mn} U_1^n U_2^m$$

The sum is finite because $U_1^N = U_2^N = \mathbf{I}$

Now the harmonics retained are finite, the space is *finite* dimen
products is consistent at a price $\boxed{U_1 U_2 = e^{\frac{2\pi i}{N}} U_2 U_1}$

This is the Fuzzy Torus originally introduce by H. W

On a fuzzy Torus there is the $U(1) \times U(1)$ transla
unchanged.

For example translation of angles α_k in the direction k are:

$$\varphi_{mn} \longrightarrow e^{i(m\alpha_1 + n\alpha_2)} \varphi_{mn}$$

There are the two derivatives:

$$\nabla_1\varphi = \sum_{m,n=1}^{N} m\varphi_{mn}U_1^m U_2^n \qquad \nabla_2\varphi = \sum_{m,n=1}^{N} n\varphi_{mn}U_1^m$$

The spectrum of the Laplacian is the same as in
nicative case, only truncated at level N

There is also a natural integral

$$\int \varphi = \mathrm{Tr}\,\varphi = \varphi_{00}$$

If N large enough, and hence the noncommutativity s
the two spaces probed by a low energy theory wo
same

Another example is the fuzzy sphere introduced by

$$[X_i, X_j] = i\frac{r}{\sqrt{N(N+1)}}\varepsilon_{ijk}X_k$$

choosing

$$X_i = \frac{r}{\sqrt{N(N+1)}}L_i$$

with the L's the usual angular momentum operators in the $N(N+1)$ repres

The Casimir of the representation becomes the con

$$X_1^2 + X_2^2 + X_3^2 = r^2$$

So that the X's define an approximation of the sp

The algebra is finite dimensional, but rotations act
as the usual three derivations (which are the X's th

Again the price we have to pay is the noncommuta
algebra

The algebra of rotations act in a natural way so t
retained all symmetries of the theory

Also in this case the Laplacian is the same as the
sphere, but is truncated

Let me now introduce the fuzzy disc

work in collaboration with P. Vitale & A. Zampini

Consider a function on the plane with its Taylor exp

$$\varphi(\bar{z}, z) = \sum_{m,n=0}^{\infty} \varphi^{\mathsf{Tay}}_{mn} \bar{z}^m z^n$$

Now "quantize" the plane, using a quantity θ ana
and associate to a function the operator

$z \longrightarrow a$ $\bar{z} \longrightarrow a^\dagger$

for convenience we choose the slightly unusual normalization

$[a, a^\dagger] = \theta$

We thus have a way to associate operators to funct

$$\Omega_\theta(\varphi) := \widehat{\varphi} = \sum_{m,n=0}^{\infty} \varphi_{mn}^{\mathsf{Tay}} a^{\dagger m} a^n$$

This is a variant of the Weyl map used to define the Moyal product

Ω_θ has an inverse expressed using *coherent* states:

$$\Omega_\theta^{-1}(\widehat{\varphi}) = \varphi(\bar{z}, z) = \langle z | \, \widehat{\varphi} \, | z \rangle$$

We can express operators with a density matrix not

$$\hat{\varphi} = \sum_{m,n=0}^{\infty} \varphi_{mn} |m\rangle \langle n|$$

With $|n\rangle$ eigenvectors of the number operator N $=$

The density matrix basis has a very simple multiplic

$$|m\rangle \langle n| p\rangle \langle q| = \delta_{np} |m\rangle \langle q|$$

The analog of the Taylor expansion in terms of th
φ_{nm} is

$$\varphi(\bar{z}, z) = e^{-\frac{|z|^2}{\theta}} \sum_{m,n=0}^{\infty} \varphi_{mn} \frac{\bar{z}^m z^n}{\sqrt{n!m!\theta^{m+n}}}$$

We have implicitly defined a noncommutative $*$ product on th

$$\left(\varphi * \varphi'\right)(\bar{z}, z) = \Omega^{-1}\left(\Omega(\varphi)\,\Omega(\varphi')\right)$$

$$z * \bar{z} - \bar{z} * z = [z, \bar{z}]_* = \theta$$

In the density matrix basis this product is the usual row by product

$$\left(\varphi * \varphi'\right)_{mn} = \sum_{k=1}^{\infty} \varphi_{mk}\varphi'_{kn}$$

Also

$$\int dz d\bar{z}\, \varphi(\bar{z}, z) = \frac{1}{2\pi}\,\mathrm{Tr}\,\widehat{\varphi} = \frac{1}{2\pi}\sum_{n=0}^{\infty} \varphi_{nn}$$

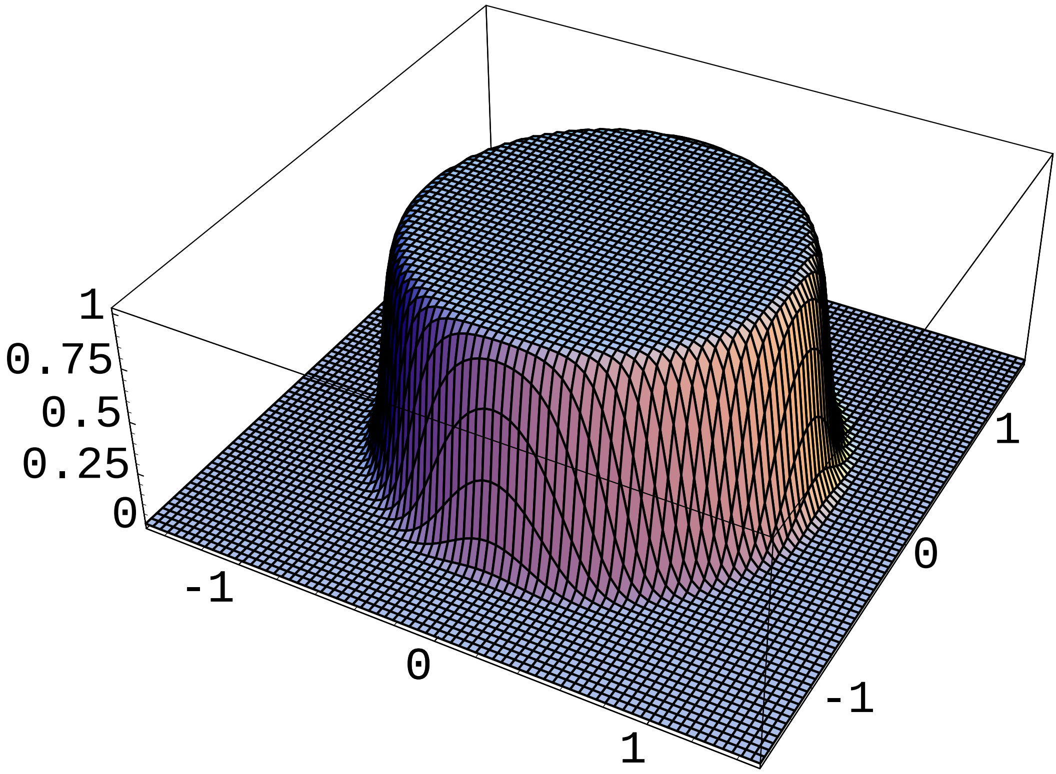
Since we reduced the product to a matrix product
that we can consider the subalgebra obtained tru
expansion to a finite N

It is the algebra obtained with the projector

$$P_\theta^N = \sum_{n=0}^{N} \langle z|\, n\rangle \langle n|\, z\rangle = \sum_{n=0}^{N} \frac{r^{2n}}{n!\theta^n} e^{-\frac{r^2}{\theta}}$$

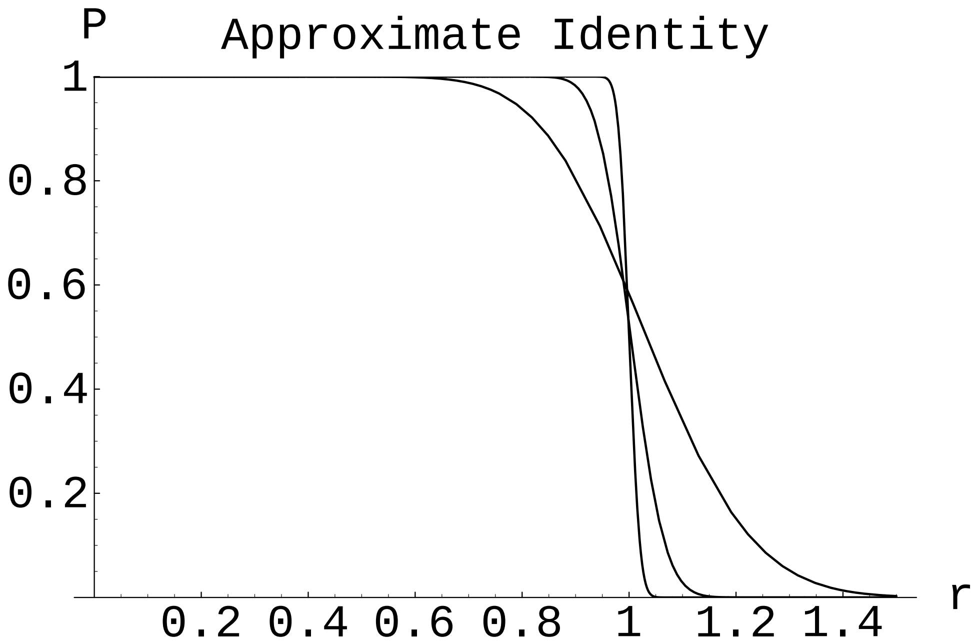
The subalgebra is clearly isomorphic $N \times N$ matrices

What sort of functions are there in this algebra?



The function P_θ^N for $N = 10^2$, $\theta = 1/N$.

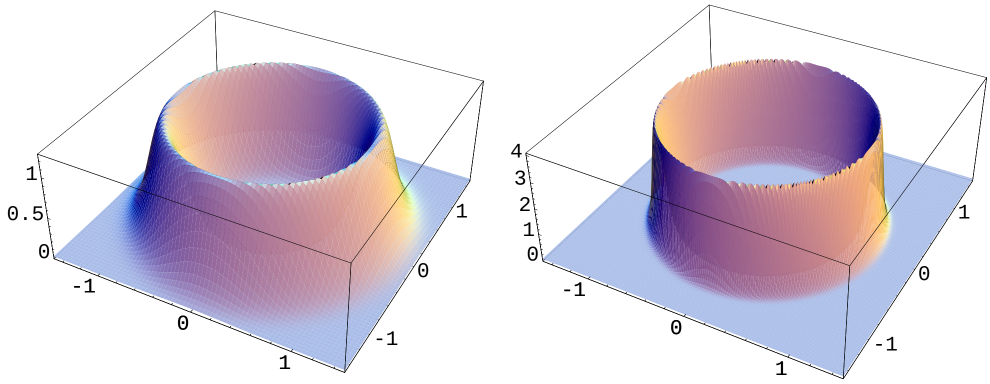
The disc becomes sharper as N increases, keeping



Profile of the spherically symmetric function P_θ^N *fo*
$R^2 = N\theta = 1$ *and* $N = 10, 10^2, 10^3$. *As* N *increa*
becomes sharper.

The subalgebra defined by P_θ^N is therefore made
which have mostly support on the disc of radius $R =$
sharply defined as N increases

We also have edge states



The edge states $\langle z|N\rangle\langle N|z\rangle$ *for* $N = 10$ *and* $N = 10$

Rotations are still there, obtained just multiplying the
by a phase.

Still I need more to convince you that this matrix al
do with the disc.

Derivatives and Laplacians

The starting point to define the matrix equivalent o
tions is:

$$\partial_z \varphi = \frac{1}{\theta} \langle z| \left[a^\dagger, \Omega(\varphi)\right] |z\rangle$$

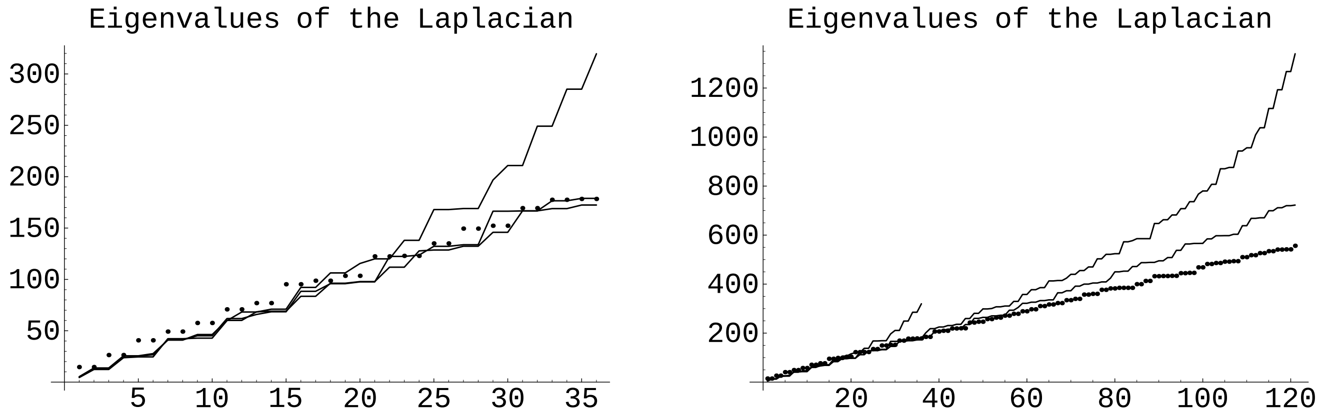
$$\partial_{\bar{z}} \varphi = \frac{1}{\theta} \langle z| \left[a, \Omega(\varphi)\right] |z\rangle$$

Note that $\partial_z(P_\theta^N * \varphi * P_\theta^N) \neq P_\theta^N * (\partial_z \varphi) * P_\theta^N$ but
they coincide (and the latter is simpler to implement for m

Since $\boxed{a}$ and $\boxed{a^\dagger}$ are still infinite matrices, $\boxed{\hat{\partial}_z \hat{\varphi}}$ a
$\boxed{N+1 \times N+1}$ matrices

The fact that functions and "form" live in different space is st

The laplacian is a finite operator and we can calculate its ei
compare them with the eigenvalues of the Laplacian on the d



The first eigenvalues of the laplacian on the disc (dots) and the *(solid lines) for $N = 5, 10, 15$. The lines corresponding to the* *be distinguished because the agreement with the exact cas* *N grows. In the figure on the right the curve which interru* *corresponding to $N = 5$, for which there are only 36 eigenvalu*

Massless Scalar Field Theory

The setting is ready to solve nonperturbatively the
path integral formalism

To a an action of the type:

$$S = \int d^2 z \varphi \nabla^2 \varphi + \frac{m^2}{2} \varphi^2$$

We associate the fuzzy action

$$S_\theta^2 = \frac{1}{\pi} \operatorname{Tr} \widehat{\varphi} \hat{\nabla}^2 \widehat{\varphi} \frac{m^2}{2} \widehat{\varphi}^2 + V(\widehat{\varphi})$$

A field theory can be solved with a path integral M
other techniques

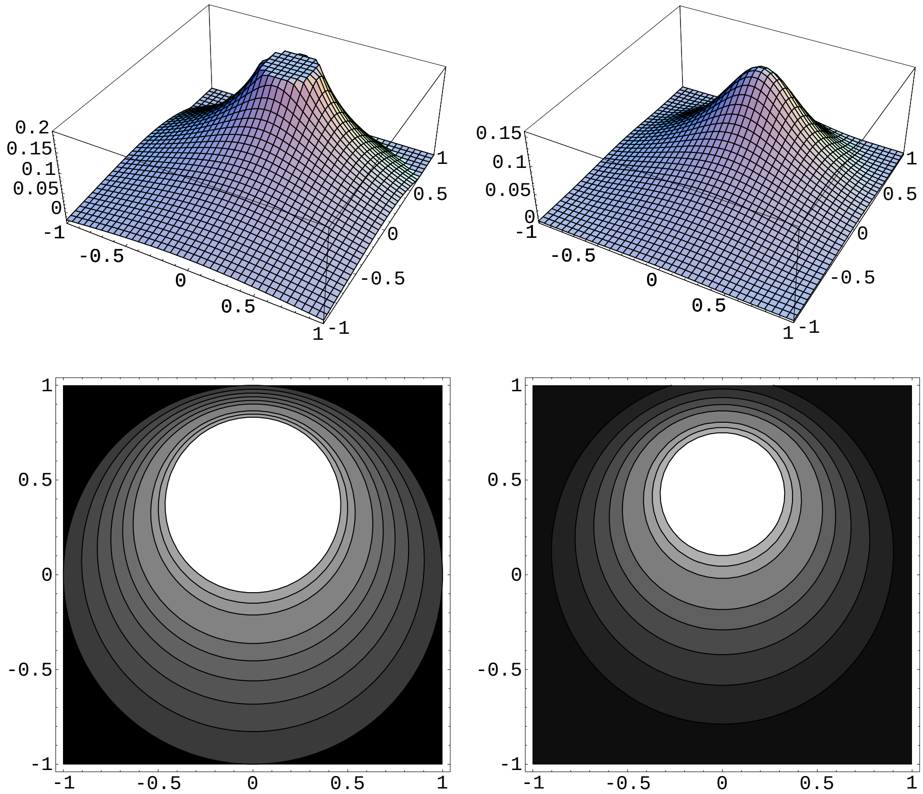
A particularly simple case is when the theory is free a
then

$$\langle\varphi(\bar{z},z)\varphi(\bar{z}',z')\rangle = G(z,z')$$

The path integral gives just the inverse of the Lapla

$$G_\theta^{(N)}(z,z') = \sum_{m,n,p,q=1}^{N} \frac{e^{-\frac{|z|^2+|z'|^2}{\theta}}(\hat{\nabla}^{-2})_{mnpq}\bar{z}^p z^q z'^m \bar{z}'^n}{\sqrt{p!q!m!n!\theta^{m+n+p+q}}}$$

We can therefore compare the Green's functions



Comparison of the 3D and contour plot of $G(z, z')$ *for* $z = 0 + 1/2i$ *as*

The fuzzy case (right) is with $N = 15$

Conclusions

Using Noncommutative Geometry we have describe
proximations which could be used to approximate fi

But I would like to add another possible and interesting aspect

If we are willing to give up commutativity then w
possibility that spacetime may be represented by ma
than continuous functions, with far fewer degrees o

All this still retaining the fundamental symmetries o