Approximate angular diameter distance in a locally inhomogeneous universe with nonzero cosmological constant

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1 Appendix

As was already noted the equation for the angular diameter distance in a locally non homogeneous Friedman-Robertson-Walker universe with non zero cosmological constant can be reduced to the Heun equation (see Eq. 28) (Heun, 1889). The Heun differential equation generalizes the Gauss hypergeometric equation, it has one more finite regular singular point (see Ince, 1964). This equation often appears in physical problems in particular in studying of diffusion, wave propagation, heat and mass transfer, and magnetohydrodynamics (see Ronveaux, 1995). In the general form the Heun equation can be written as

$$\begin{aligned}&\frac{d^2H}{dy^2} + \left(\frac{a_1}{y-y_1} + \frac{a_2}{y-y_2} + \frac{a_3}{y-y_3}\right)\frac{dH}{dy} + \\ &\frac{(a_4 y - q)}{(y-y_1)(y-y_2)(y-y_3)}H = 0,\end{aligned} \tag{1}$$

where a_1, a_2, a_3, a_4, q, y_1, y_2 and y_3 are constants, or in a canonical form as (Bateman & Erdélyi 1955)

$$\begin{aligned}&x(x-1)(x-a)\frac{d^2H}{dx^2} + \\ &\left\{(\alpha+\beta+1)x^2 - [\alpha+\beta+1+a(\gamma+\delta)-\delta]\,x + a\gamma\right\}\frac{dH}{dx} +\end{aligned}$$

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$$(\alpha\beta x - q)H = 0, \tag{2}$$

where α, β, γ, δ, a, and q are constants.

When γ is not an integer, the general solution of the Heun equation can be written as:

$$\begin{aligned} H(x) = {} & C_1 F(a, q, \alpha, \beta, \gamma, \delta, x) + \\ & C_2 |x|^{1-\gamma} F(a, q_1, \alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, \delta, x), \end{aligned} \tag{3}$$

where C_1 and C_2 are constants and $q_1 = q + (\alpha - \gamma + 1)(\beta - \gamma + 1) - \alpha\beta + \delta(\gamma - 1)$.

Following our success in finding an approximate solution of the equation for angular diameter distance we would like to present an approximate solution of the Heun equation. It has the form:

$$H(x) = \frac{B\frac{x}{x_0}\left(\frac{x}{x_0} - 1\right)}{\sqrt{\left(1 + d_2\left(\frac{x}{x_0} - 1\right) + d_3\left(\frac{x}{x_0} - 1\right)^2\right)^2 + d_1\left(\frac{x}{x_0} - 1\right)^2}}, \tag{4}$$

where the parameter x_0 represents the point where the initial conditions are specified and d_1, d_2, d_3, B are constants. For the Heun equation we adopt the following initial conditions [5] :

$$H[x_0] = 0, \tag{5}$$

$$\frac{dH}{dx}|_{x_0} = \frac{B}{x_0}. \tag{6}$$

Please note that the form (4) automatically satisfies the imposed initial conditions.

To express the constants d_1, d_2 and d_3 in terms of α, β, γ, δ, a, and q, it is necessary to insert the proposed form of $H(x)$ into the canonical Heun equation (2) and require that it be satisfied to the highest order.We We obtained that

$$d_1 = \frac{P_3(a)}{b} \tag{7}$$

where

[5] To get the general initial conditions i.e. $H(x_0) = A$ it is enough to add to the (4) a constant $H + A$.

$$b = 6\,(a-x_0)^2\,(-1+x_0)^2\,[(1-x_0)\,x_0\,(3+\alpha+\beta) \\ +\,a\,(-1+x_0)\,(2+\gamma) - x_0\,\delta + a\,x_0\,\delta] \tag{8}$$

and $P_3(a)$ is a polynomial of 3-th degree in a: more exactly

$$P_3(a) = \mathbf{P_{33}}a^3 + \mathbf{P_{32}}a^3 + \mathbf{P_{31}}a + \mathbf{P_{30}} \tag{9}$$

where

$$\begin{aligned}
\mathbf{P_{30}} = {} & x_0^6\,(1+\alpha+\beta)\,(-1+\alpha+\beta+2\,\alpha\,\beta) \\
& -{x_0}^5\,\Big[-3+2\,q\,(\alpha+\beta)+\alpha^2\,(3+4\,\beta)+\beta\,(3\,\beta-\delta)+\delta \\
& +\,\alpha\,(4\,\beta\,(3+\beta)-(1+2\,\beta)\,\delta)] \\
& +\,{x_0}^4\,\Big[\alpha^2\,(3+2\,\beta)+\alpha\,(2\,\beta\,(5+\beta-\delta)-3\,\delta)+q\,(2+4\,\alpha+4\,\beta-2\,\delta) \\
& +\,3\,(-1+\beta)\,(1+\beta-\delta)] \\
& -\,{x_0}^3\,[(1+\alpha+\beta-\delta)\,(-1+2\,q+\alpha+\beta-\delta)]
\end{aligned} \tag{10}$$

$$\begin{aligned}
\mathbf{P_{31}} = {} & {x_0}^2\,[(2+2\,\alpha+2\,\beta+\gamma-2\,\delta)\,(-1+\alpha+\beta-\delta) \\
& +\,2\,q\,(2+\alpha+\beta+\gamma-\delta)] \\
& -\,{x_0}^3\,\Big[2\,\alpha^2\,(3+\beta)+4\,q\,(1+\alpha+\beta+\gamma)+3\,(-1+\beta)\,(2+2\,\beta+\gamma) \\
& +\,\alpha\,(3\,\gamma+2\,\beta\,(10+\beta+\gamma-\delta)-5\,\delta)-(-7+5\,\beta+\gamma)\,\delta-\delta^2\Big] \\
& +\,{x_0}^4\,\Big[\alpha^2\,(6+4\,\beta)+3\,(-1+\beta)\,(2+2\,\beta+\gamma)+\alpha\,(3\,\gamma+4\,\beta\,(6+\beta+\gamma)) \\
& +\,2\,q\,(\alpha+\beta+\gamma+\delta)] \\
& -\,{x_0}^5\,\Big[2\,\alpha^2\,(1+\beta)+(-1+\beta)\,(2+2\,\beta+\gamma+\delta)+\alpha\,(\gamma+\delta+2\beta\,(4+\beta+\gamma+\delta))\Big]
\end{aligned} \tag{11}$$

$$\begin{aligned}
\mathbf{P_{32}} = {} & -x_0\,[2\,q\,(1+\gamma)+3\,\gamma\,(-1+\alpha+\beta-\delta)] \\
& +\,x_0^2\,\Big[9\,(-1+\beta)\,\gamma+4\,(1+\beta-\gamma)\,\delta-4\,\delta^2+2\,q\,(1+2\,\gamma+\delta) \\
& +\,\alpha\,(9\,\gamma+2\,\beta\,(2+\gamma)+4\,\delta)] \\
& -\,{x_0}^3\,[\gamma\,(-9+2\,q+9\,\beta-\delta)+(-5+2\,q+7\,\beta-\delta)\,\delta+ \\
& +\,\alpha\,(9\,\gamma+7\,\delta+2\,\beta\,(3+2\,\gamma+\delta))] \\
& +\,{x_0}^4\,[3\,(-1+\beta)\,(\gamma+\delta)+\alpha\,(3\,(\gamma+\delta)+2\,\beta\,(1+\gamma+\delta))]
\end{aligned} \tag{12}$$

$$\begin{aligned}
\mathbf{P_{33}} = {} & (-2+\gamma)\,\gamma-3\,x_0\,\gamma\,(-2+\gamma+\delta)+{x_0}^2\,(-2+\gamma+\delta)\,(3\,\gamma+2\,\delta) \\
& -\,{x_0}^3\,(-2+\gamma+\delta)\,(\gamma+\delta)\,.
\end{aligned} \tag{13}$$

Analogously we reconstruct d_2:

$$d_2 = \frac{x_0^2\,(-3-\alpha-\beta)+a\,(-2-\gamma)}{2\,(a-x_0)\,(-1+x_0)} + \frac{x_0\,(3+\alpha+\beta-\delta+a\,(2+\gamma+\delta))}{2\,(a-x_0)\,(-1+x_0)} \tag{14}$$

and finally d_3:

$$d_3 = \frac{\Delta}{\Phi}, \tag{15}$$

where

$$\Phi = {x_0}^2\ (-3 - \alpha - \beta) - a\ (2 + \gamma) + x_0\ (3 + \alpha + \beta + a\ (2 + \gamma) \tag{16}$$
$$- \delta + a\,\delta)$$
$$\Delta = \mathbf{l_{33}} a^3 + \mathbf{l_{32}} a^2 + \mathbf{l_{31}} c + \mathbf{l_{30}} \tag{17}$$

where $\mathbf{l_{30}}$ $\mathbf{l_{31}}\,\mathbf{l_{32}}\,\mathbf{l_{33}}$ depend on $\alpha, \beta, \gamma, \delta, q$ and the initial point x_0. Actually

$$\begin{aligned}
\mathbf{l_{30}} = & -{x_0}^6\ (2 + \alpha + \beta)\ \left(7 + \alpha\ (8 + \alpha) + 8\,\beta + 6\,\alpha\,\beta + \beta^2\right) \\
& + {x_0}^5\ \left[3\,\alpha^3 + 3\ (1 + \beta)\ (2 + \beta)\ (7 + \beta) + q\ (6 + 4\,\alpha + 4\,\beta)\right. \\
& + \alpha\ (69 + \beta\ (78 + 17\,\beta - 10\,\delta) - 17\,\delta) + \alpha^2\ (30 + 17\,\beta - 3\,\delta) \\
& - (16 + \beta\ (17 + 3\,\beta))\ \delta] \\
& - x_0^4\ \left[42 + 3\,\alpha^3 + 69\,\beta + \alpha^2\ (30 + 13\,\beta - 6\,\delta) + 2\,q\ (7 + 4\,\alpha + 4\,\beta - 2\,\delta)\right. \\
& + (3\,\beta\ (10 + \beta - \delta) - 7\,\delta)\ (\beta - \delta) - 37\,\delta + \alpha\ \left(69 + 13\,\beta^2 + \beta\ (70 - 16\,\delta)\right. \\
& -\ \delta\ (-37 + 3\,\delta))] \\
& + {x_0}^3\ (4\,q + (1 + \alpha + \beta - \delta)\ (7 + \alpha + \beta - \delta))\ (2 + \alpha + \beta - \delta)
\end{aligned} \tag{18}$$

Analogously for $\mathbf{l_{31}}$ we obtained

$$\begin{aligned}
\mathbf{l_{31}} = & \ {x_0}^5\ \left[\alpha^2\ (10 + 4\,\beta + 3\,\gamma + 3\,\delta) + 2\ (13 + 8\,\gamma + 8\,\delta)\right. \\
& + \beta\ (36 + 17\,\gamma + 17\,\delta + \beta\ (10 + 3\,\gamma + 3\,\delta)) + \alpha\ (36 + 17\,\gamma + 17\,\delta + \\
& +2\,\beta\ (17 + 2\,\beta + 5\,\gamma + 5\,\delta))] \\
& +{x_0}^4\ \left[-78 - 10\,q + \beta^2\ (-30 - 9\,\gamma - 6\,\delta) + \alpha^2\ (-30 - 8\,\beta - 9\,\gamma - 6\,\delta)\right. \\
& + (-14 - 4\,q)\ \delta + 14\,\delta^2 + \gamma\ (-48 - 4\,q + 14\,\delta) + \beta\ (-108 - 4\,q - 20\,\delta \\
& \left. +\ 6\,\delta^2 + \gamma\ (-51 + 6\,\delta)\right) + \alpha\ \left(-108 - 4\,q - 8\,\beta^2 + \beta\ (-92 - 26\,\gamma - 12\,\delta)\right. \\
& \left.\left. -20\,\delta + 6\,\delta^2 + \gamma\ (-51 + 6\,\delta)\right)\right] \\
& +{x_0}^3\ [78 + 48\,\gamma + 8\,q\ (3 + \alpha + \beta + \gamma) + 3\,\beta\ (36 + 17\,\gamma + \beta\ (10 + 3\,\gamma)) \\
& + (-32 + \beta\ (-17 + 3\,\beta - 12\,\gamma) - 31\,\gamma)\ \delta - (13 + 6\,\beta - 3\,\gamma)\ \delta^2 + 3\,\delta^3 \\
& +\alpha^2\ (30 + 4\,\beta + 9\,\gamma + 3\,\delta) + \alpha\ \left(108 + 4\,\beta^2 + 51\,\gamma - 17\,\delta - 6\,\delta\ (2\,\gamma + \delta)\right. \\
& +2\,\beta\ (39 + 11\,\gamma + \delta))] \\
& -{x_0}^2\ [-6 + 36\,\alpha + 36\,\beta + 16\,\gamma + 2\,q\ (7 + 2\,\alpha + 2\,\beta + 2\,\gamma - 2\,\delta) \\
& -36\,\delta + (\alpha + \beta - \delta)\ (17\,\gamma + \alpha\ (10 + 3\,\gamma) + \beta\ (10 + 3\,\gamma) - 10\,\delta - 3\,\gamma\,\delta)]
\end{aligned} \tag{19}$$

and for $\mathbf{l_{32}}$:

$$\begin{aligned}
\mathbf{l_{32}} = {} & x_0\,[12 + 12\,\alpha + 12\,\beta + 23\,\gamma + q\,(6 + 4\,\gamma) - 12\,\delta \\
& +\gamma\,(7\,\gamma + \alpha\,(17 + 3\,\gamma) + \beta\,(17 + 3\,\gamma) - 17\,\delta - 3\,\gamma\,\delta)] \\
& +{x_0}^2\,[-3\,(12 + \gamma\,(23 + 7\,\gamma) + \beta\,(12 + \gamma\,(17 + 3\,\gamma))) \\
& -2\,(1 + \beta - \gamma)\,(10 + 3\,\gamma)\,\delta + 2\,(10 + 3\,\gamma)\,\delta^2 - 2\,q\,(5 + 4\,\gamma + 2\,\delta) \\
& -\alpha\,\left(36 + 51\,\gamma + 9\,\gamma^2 + 4\,\beta\,(2 + \gamma) + 20\,\delta + 6\,\gamma\,\delta\right)] \\
& +{x_0}^3\,\left[36 + 36\,\beta + 69\,\gamma + 51\,\beta\,\gamma + 3\,(7 + 3\,\beta)\,\gamma^2 + (49 + (17 - 3\,\gamma)\,\gamma\right. \\
& +\beta\,(37 + 12\,\gamma))\,\delta + (-4 + 3\,\beta - 6\,\gamma)\,\delta^2 - 3\,\delta^3 + 4\,q\,(1 + \gamma + \delta) + \\
& \alpha\,(36 + 51\,\gamma + 37\,\delta + 3\,(\gamma + \delta)\,(3\,\gamma + \delta) + 2\,\beta\,(7 + 4\,\gamma + 2\,\delta))] \\
& -{x_0}^4\,[12 + 12\,\beta + 23\,\gamma + 23\,\delta + (\gamma + \delta)\,(7\,(\gamma + \delta) + \beta\,(17 + 3\,\gamma + 3\,\delta)) \\
& +\alpha\,\left(12 + 17\,\gamma + 17\,\delta + 3\,(\gamma + \delta)^2 + \beta\,(6 + 4\,\gamma + 4\,\delta)\right)]
\end{aligned} \tag{20}$$

Finally for $\mathbf{l_{33}}$ we obtained:

$$\begin{aligned}
\mathbf{l_{33}} = {} & -\gamma\,\left(6 + 7\,\gamma + \gamma^2\right) + x_0\,[3\,\gamma\,(1 + \gamma)\,(6 + \gamma) + (12 + \gamma\,(17 + 3\,\gamma))\,\delta] \\
& -{x_0}^2\,\left[3\,\gamma\,(1 + \gamma)\,(6 + \gamma) + (16 + \gamma\,(31 + 6\,\gamma))\,\delta + (10 + 3\,\gamma)\,\delta^2\right] \\
& +{x_0}^3\,(\gamma + \delta)\,(1 + \gamma + \delta)\,(6 + \gamma + \delta)
\end{aligned} \tag{21}$$

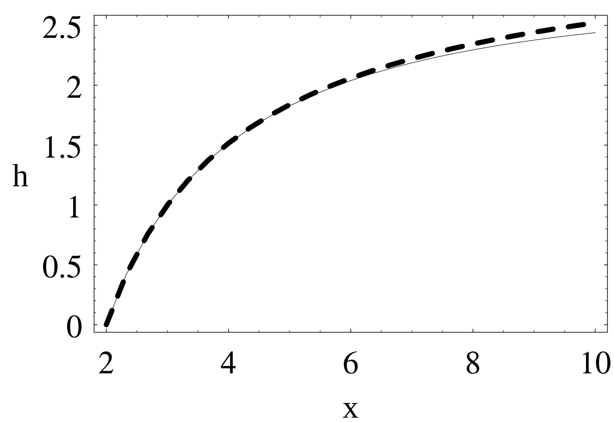

Fig. 1. Comparison of the exact (solid line) with the approximate solution (dashed line) of the Heun equation.

In Fig. 5 we compare an exact numerical solution of the Heun equation with the approximate solution for the same values of α, β, γ, δ, a and q.

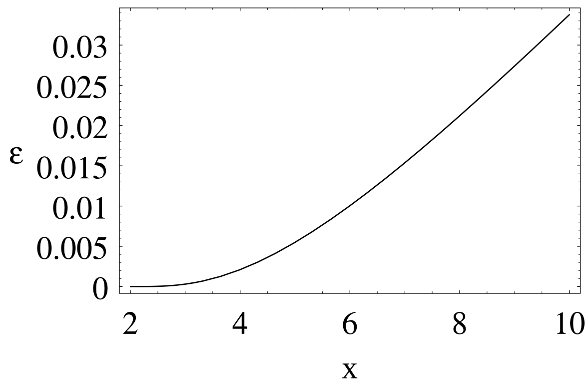

Fig. 2. Relative error of the approximate solution of the Heun equation as a function of x.

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